GRAPHICAL AND TABULAR METHODS IN CRYSTALLOGRAPHY

T. V. BARKER

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GRAPHICAL AND TABULAR METHODS IN CRYSTALLOGRAPHY

AS THE FOUNDATION OF A NEW SYSTEM OF PRACTICE

WITH A MULTIPLE TANGENT TABLE AND A
5-FIGURE TABLE OF NATURAL COTANGENTS

BY

T. V. BARKER

University Lecturer in Chemical Crystallography and Fellow of
Brasenose College, Oxford
Honorary Associate of the School of Mines, Petrograd

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PREFACE

'The advantages of graphical methods over numerical calculations are numerous, and are fully appreciated by engineers and others who deal extensively with measurements and practical results derived therefrom. . . . In the majority of cases, numerical calculations are laborious, while graphical solutions appeal to one like pictures which, to a certain extent, tell their own story.'

S. L. PENFIELD (1901).

THE general omission from the list of contents of problems belonging to the two provinces of Structure and Optics is enough to indicate that this is not the complete treatise a reader might well expect from the principal title. Even in the geometrical domain there has been no attempt to bring together all the methods which are to be found in a century of original literature. Many graphical solutions are omitted, as being of purely theoretical interest; alternative solutions are generally avoided, as liable to create a state of bewilderment; and rigid proofs (even in the rare case of a new construction or novel property) are dispensed with on grounds of economy. As imperfectly indicated by the title, the main purposes of the book are to provide the researcher with a select collection of exact graphical methods, which personal experience has proved to be both accurate and time-saving; to discuss the relation of these methods to formal processes of computation; and finally, to outline a new system of practice. The various problems of goniometry it is proposed to examine in a separate monograph (almost ready for the press), while certain aspects of structure and optics will naturally come up for treatment in an eventual treatise on Crystallochemical Analysis.

So much by way of general preface. Unfortunately, a somewhat lengthy introduction would seem to be unavoidable. Graphical operations of precision cannot be easily effected without the help of special instruments, designed to obviate frequently occurring and time-consuming constructions. Many such auxiliaries (protractors, nets, three-point compasses, drawing machines of an ever increasing degree of complexity) have been recently described and vigorously pressed into service by their inventors. They will not be considered in detail here, as actual experience shows that they can be suitably united into a single instrument (the main exception being the generalised form of stereographic net, which is however only rigorously demanded by the problems of optics, crystal structure and crystallochemical analysis).

The first chapter is accordingly devoted to a brief description of this 'crystallographic protractor,' which would seem to embody the easily workable features of all its predecessors (especially those of Penfield, Fedorov and Hutchinson). The subsequent treatment conforms to the lines of the typical geometrical investigation. Some account of the principles of two and three-circle measurement and calculation is necessary to a correct understanding of this book as a whole, so it has been informally incorporated into Chapter II, dealing with the preparation of a stereographic projection from angular measurements of various kinds. The following chapter on the gnomonic projection is not unimportant since no one can hope to profit from recent improvements in technique, unless he can handle this projection as dexterously as the stereographic. Chapters IV and V are respectively devoted to the graphical determination of indices and the preparation of crystal drawings. The first five chapters and a later chapter on 'The four related projections and their crystallographic history' constitute the purely graphical portion of these pages.

Chapter VI and its appendices I—III constitute the second part of the book—that which is implied by the term 'tabular methods' of the title. The main objects are to illustrate the close relation between graphical methods and formal methods of computation and to show how one may usefully replace or facilitate the other. The chapter is not so much concerned with calculations in general as with the Millerian formulae connecting angles and indices of four tautozonal faces or coplanar axes (the more difficult treatment of the oblique spherical triangle being deferred to a later chapter). The first part deals with its most general aspect, the non-rectangular case. The view is taken that the sine formula has little if any present-day applicability, since indices can generally be more advantageously determined graphically. With regard to the converse operation, the superiority of the cotangent formula (as opposed to the 'Theta variant'), especially in the omnipresent 'harmonic case' is demonstrated and a five-figure table of natural cotangents (and tangents) is therefore included as Appendix I. The second part of the chapter deals with the special case of rectangularity, in which the trigonometrical side of the equation reduces to a ratio of two tangents. The solution of this equation for every possible case that is likely to occur in actual practice is given as a 'multiple tangent' table in Appendix III—a contrivance which almost wholly replaces *ad hoc* zonal calculations in all systems down to the orthorhombic and brings a measure of relief in the monoclinic system. The chapter also contains several new relations (new to crystallographers) of practical value which serve to extend both the general Millerian formula and the multiple tangent table to the more special angles of two-circle goniometry.

The inclusion of a list of useful formulae (Appendix II, dividing the numerical tables) will possibly add to the usefulness of the book. The same may, perhaps, be said of the numerous exercises, placed at the end of every Chapter. They all have a practical incidence, and as solutions are given (at the end of the book) in nearly every case, they may prove especially useful to a reader who is not within easy reach of occasional guidance. Most of the angular values are drawn from Goldschmidt's 'Winkeltabellen,' and in view of the accuracy of this work, I have not hesitated to copy out the answers, although these will occasionally show a difference possibly up to 2′ from the strictly worked-out result, owing to an occasional initial omission of a half-minute.

It is in the last Chapter of the book that I have ventured to outline a New System of Practical Crystallography. The classical system, as is well known, is based wholly on the stereographic projection, not so much as a quantitative agent as a general guide to the serious work of calculation. The new system, on the other hand, is erected on a combined gnomono-stereographic projection as an instrument of precision. When divested of numerous ancillaries, the simple issue between the two crystallographic systems is whether the reduplicated and subdivided gnomonic parallelogram, as measured by a simple millimetre scale, is capable of replacing a system of spherical triangles, as measured by logarithmic formulae of varying complexity; and, as I cannot assume that a reader has already studied Chapter IV, I may here explain that this issue can be translated into the question whether the index numbers of crystal faces are or are not less than seven. Now mineralogical treatises indicate that although the forms commonly developed on minerals have simple indices, yet faces having complex index-numbers are occasionally observed and must be cared for (hence, no doubt, the mineralogical origin of the classical method). But compendia which deal rather with crystals as a whole, as, *e.g.* Groth's *Chemische Krystallographie*, Vols. I—V, and its unique supplementary index, Fedorov's unpublished *Crystal Kingdom*—respectively the 'Beilstein' and the 'Richter' of Crystallography; these comprehensive works of reference demonstrate that the face with higher index-numbers than 6 (that which cannot definitely be measured by the gnomonic projection) is so rare, that the probability of its intrusion on the notice of any given crystallographer is zero. This would seem to decide the question of the relative suitability of the two systems, so far as laboratory products are concerned. The more difficult appraisal of the new system in the province of minerals is a task which can be more competently effected by specialists.

Chapter VIII, then, is devoted to a detailed consideration of the various operations, involved in the geometrical investigation of a

crystal, from the central standpoint of the new system—which is based on the principle that it should not be necessary to compute what has already been measured. This does not mean that formal computations have no useful place in the system; what it does imply is that they must not be allowed to take charge of the investigation immediately after the completion of the goniometrical measurements, and finally commit the unlucky investigator of an anorthic crystal to the direct or indirect solution of some twenty oblique spherical triangles. The greater part of the work, incidental to the construction of the new system, has as a matter of fact been devoted to a consideration of the question how far calculations may be usefully applied. A perusal of the Chapter will show that elements are retained, although this may involve the solution of as many as three spherical triangles in the case of the single-circle worker. The two-circle worker is in a more favourable position, for he can actually measure the elements of an anorthic crystal on the goniometer if he wishes. He can also usefully carry out a few simple calculations (and so also can the single-circle worker) especially designed either to facilitate the derivation of more accurate elements, or to render it unnecessary to measure so many crystals of the same substance, as was formerly the case. In various ways, which have occurred to me during the last twelve years, I have sought to improve the system as much as possible: always by substituting what appears to be complex by something simpler, no matter which be old and which be new—for new practices are not necessarily simpler than old, nor old than older. As a result, it seems possible to reduce the labour, incidental to the accurate determination of geometrical constants, to less than one-half of that formerly demanded. This of course is an average figure, for the efficiency-ratio naturally depends somewhat on the type of goniometer and varies from system to system, most time being saved in the anorthic and none at all in the cubic. It also fluctuates from substance to substance within the same system; the more highly developed the crystal the relatively greater the saving of time. The new method or system will naturally be of greatest value to those who specialise in the description of new substances, or who look forward to a time when crystalline substances will be identified as often by measurement as by analysis. It is especially the crystallographer who, so to speak, would travel towards this chemical horizon, that must throw off the incubus of the oblique case, at any rate, of plane and spherical trigonometry.

I have, fortunately, to put on record a lengthy series of acknowledgments. Much more than I can express I owe to the kindness of the late Dr. Heberden and the present Principal and Fellows of Brasenose College, who have found it possible to endow the study of a science which is, perhaps, too rarely fostered. Of my chemical teacher, Mr.

J. E. Marsh, and my earliest teachers in crystallography, Sir Henry A. Miers and his Oxford successor, Professor H. L. Bowman, I would say that their stream of encouragement shows no sign of diminution with lapse of years; and of my later teachers, Professor Groth, the late Professor Fedorov and Professor Nikitin, that the memory of their kind favours is ineffaceable. I also owe much to frequent discussions of crystallographic problems with Dr. J. Drugman. Professor Hilton has given me much help in those mathematical aspects of the science he has made his own. I am also indebted to Dr. G. T. Prior and Dr. G. F. Herbert Smith, of the British Museum (Natural History) for permission to use the three-circle goniometer which the latter invented. Some of the calculations for the multiple tangent table were undertaken by my wife; others by Mr. R. C. Spiller, who has also assisted with the drawings and in many other ways. My thanks are also due to my Publishers, and their Printers, for their unfailing courtesy during the process of publication; and also to Messrs. J. H. Steward, Ltd., for the ready way in which they undertook to place the Crystallographic Protractor within the reach of those who may find a use for it. In this connection it is relevant to add that the whole of the drawings (as also at least another hundred) have been effected by its help—a service which has thoroughly tested its workmanlike capabilities.

T.V.B.

MINERALOGICAL DEPARTMENT,
UNIVERSITY MUSEUM,
OXFORD.

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EXPLANATORY NOTE.

The terms $(\phi_2 - \phi_1)$ and $(\rho_2 - x)$ of the two-circle transformation formula of p. 134 merely imply *differences* (the smaller always to be subtracted from the larger), but in every other respect there must of course be a due regard to algebraic signs. Thus, if the difference $(\phi_2 - \phi_1)$, or $(\phi_1 - \phi_2)$ as the case may be, has a greater value than 90°, the subsequent substitution of its supplementary value implies a reversal of sign in cosine or tangent. It need scarcely be added that if one of the original ϕ-values is negative it should be first turned into the alternative positive form by the addition of 360° (*cf.* Fig. 15, p. 12).

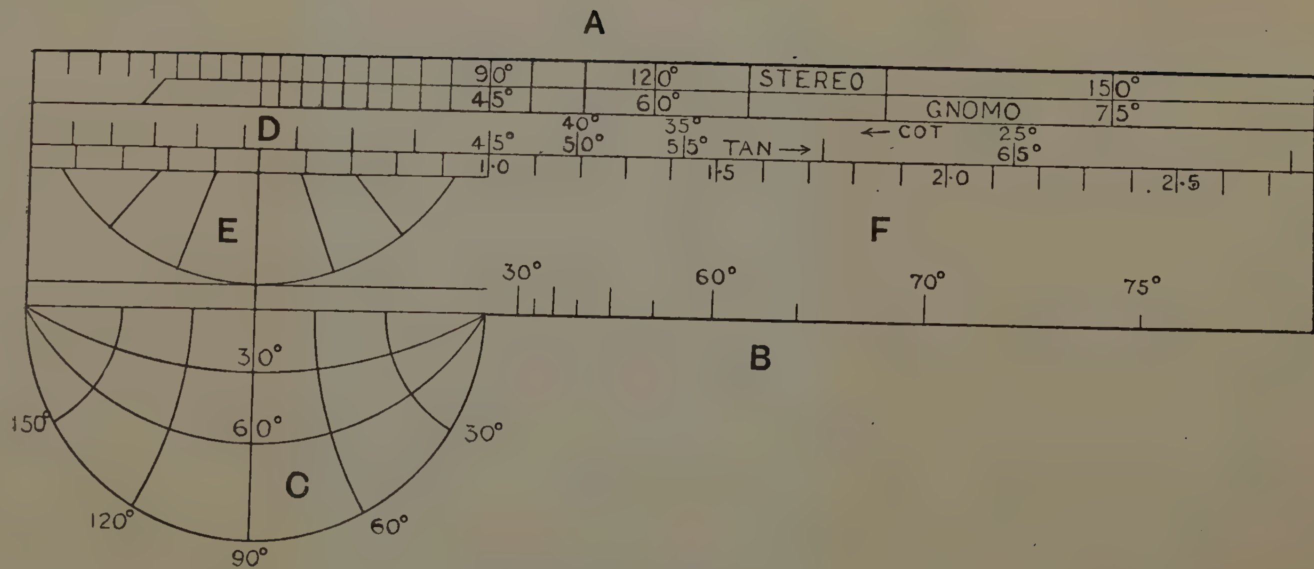


FIG. 1. THE CRYSTALLOGRAPHIC PROTRACTOR ($\frac{2}{3}$ NATURAL SIZE).

GRAPHICAL AND TABULAR METHODS OF CRYSTALLOGRAPHY

CHAPTER I.

THE CRYSTALLOGRAPHIC PROTRACTOR (AND OTHER AUXILARIES).

'But there is no doubt that the most important practical question is how to attain accuracy by means of the simplest tools. Only when the desired degree of accuracy is not thus obtainable must we turn to less simple methods' [Russian].

E. S. FEDOROV (1905).

'Any novelty there may be in the device is to be sought therefore in the convenient arrangement of its parts, and in the ease with which it may be constructed and used, rather than in the principles which it embodies.' A. HUTCHINSON (1908).

The protractor is made of transparent celluloid in one size only, the primitive circle having a radius of 5 cms. A greater radius is incompatible with the convenient use of the gnomonic projection, and actual work with the protractor proves that the same radius of 5 cms. is sufficiently large for general stereographic purposes. As judged by corresponding results computed trigonometrically, the graphical error in the solution of any problem never exceeds $\frac{1}{2}^\circ$, except under rare, unfavourable circumstances. This means that for many practical purposes the protractor is an efficient substitute for logarithmic calculations.

A general but much simplified representation (about two-thirds of the natural size) is given in Fig. 1, and (about one-third of the natural size) in Fig. I*a* (p. 3). The various parts are labelled *A, B, C, D, E* and *F*. They will now be briefly described.

A.—The far edge *A* is the main stereographic scale—perhaps the most important part of any protractor. The nature and extent of this scale is illustrated by Fig. 2, which represents a vertical section of the sphere of projection, *N* and *S* being the North and South Poles and *CD* the trace of the equatorial plane (the plane of projection). Any pole on the sphere a', say 30° distant from *N*, is projected to the point *a* (the distance *Oa* being given by the well known formula, $Oa = r \tan (30^\circ/2)$; again, a pole b', 120° distant from *N*, is projected to the point *b* (necessarily beyond the confines of the equatorial circle *CD* since the angle is greater than 90°). The main scale '*A*' of the protractor represents the various projected points, ranging from *C* on

the left to a point E 154° distant on the right, *the graduations being given every* 2°. As will be explained on p. 22, the stereographic scale is simultaneously a gnomonic scale, each 2° of the stereographic graduation being equal to 1° of the gnomonic scale. The latter

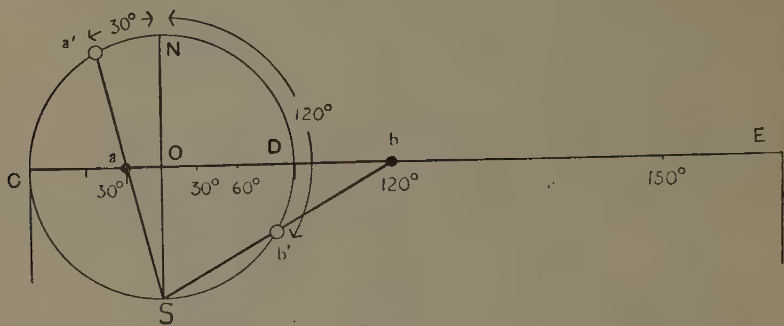


FIG. 2.

therefore ranges from 45° on the left to 77° on the right. The gnomonic numeration[1] is only engraved on the portion to the right of the centre—the numbers being placed immediately behind the stereographic. Scale '*A*' corresponds to the scale marked on both edges of Hutchinson's protractor, to Fedorov's main scale, and also to Penfield's scale No. 3 (which was designed for a radius of 7 cms.); its numerous uses will be described later.

B.—The near edge *B* carries a scale of centres of projected 'small' circles, each of which represents the locus of points $x°$ from any given pole on the stereographic primitive. The method of using this scale is described on p. 6. This scale corresponds to Fedorov's subsidiary scale[2] and to Penfield's scale No. 1. The graduations are given every 5° within the limits 10°—40°, and every 1° within the limits 40°—77°. Intermediate values are readily estimated.

C.—This part, on which are engraved meridian and small circles every 5° and also peripheral graduations every 1°, is similar to Penfield's 'stereographic protractor III.' Although salient to the main body, it causes no working inconvenience; moreover, the fact that it is part of the protractor (or, in other words, that there is *one* protractor

[1] It may be of interest to mention that the gnomonic scale on a crystallographic protractor is nothing more nor less than the scale along the upper edge of the ordinary Service protractor. The primary use of the latter is of course to measure or lay off angles. This 'rectangular' form of instrument appears to have come into use in the early part of the eighteenth century.

[2] The arrangement of scales *A* and *B* was suggested by a pair of magnificent steel protractors which were executed for the writer by Petermann, of Petrograd (in 1908), from a design supplied by the late Professor Fedorov. These two protractors (for a circle of 10 cms. radius) are possibly the most accurate extant. If used carefully they admit of the graphical solution of problems with an accuracy of 5′. Their present cost would be about £15 each.

and not several separated parts) constitutes one of the principal advantages of the instrument. The offices of this part are: (1) to lay off angles on the primitive; (2) to test zonality of any triad of face-poles;

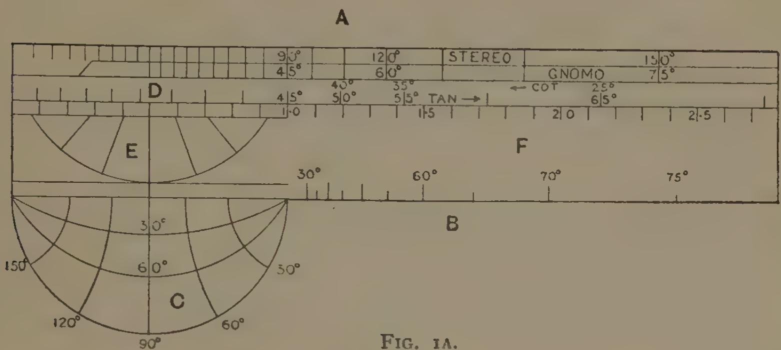


FIG. 1A.

and (3) to measure angles in projection between face-poles or zone-circles—with an error rarely if ever exceeding $\frac{1}{2}°$.

D.—This is a tangent (or co-tangent) scale for the reversible transformation: angles $\rightleftarrows$ axial ratios in the rectangular systems. The graduations give two decimal places and the third may be closely estimated. Thus, as read from the scale, tan $35° = 0·699$ (as compared with the true value 0·700).

E.—This is designed to facilitate the 'development of a zone,' as also the graphical determination of indices, by a method independently introduced by Fedorov (1887) and by Moses and Rogers (1902).

F.—This part is occupied by two Millerian formulae. The author's recognition that these formulae are obeyed by the 'phi-angles' of two-circle goniometry extends their application to the oblique systems.

The Flexible Ruler.—Some meridian and small circles have too long a radius to be drawn by ordinary compasses, even when fitted with the leg-extension, and some form of flexible ruler is indispensable to accurate work with the stereographic projection. A suitable 'arc-ruler' for crystallographic purposes was first constructed by G. Wulff (*Zeitsch. Kryst. Min.*, 1893, **21**, 253), and this has been improved by E. S. Fedorov (the latest form being described and figured in *The New Geometry as the Foundation of Drawing* [Russian], 1907, p. 107); also by S. L. Penfield, who preferred a wooden instrument (*Amer. J. Sci.*, 1901, **11**, 138); and finally by G. F. H. Smith (*Min. Mag.*, 1913, **17**, 46).

Stereographic Nets.—As these are not so useful in single-circle as in two- and three-circle work, it seems better to postpone their description to p. 14. They are also indispensable in optical and structural problems, but these lie outside the scope of this book.

CHAPTER II.

PREPARATION OF A STEREOGRAPHIC PROJECTION.

'The first great improvement in the methods of crystallography, after its establishment as a science, was undoubtedly made by Mohs and Weiss, independently of each other, in substituting axes for hypothetical integrant molecules, in the enunciation of the geometric laws discovered by Hauy. Next to this in importance was the graphic method invented by Neumann, and described in his Beiträge zur Krystallonomie.'

W. H. MILLER (1859).

The recent introduction of two- and three-circle goniometers has to be taken into account by any writer on graphical methods, for the obvious reason that goniometrical measurements constitute a great part of his raw material. In the present instance they constitute the whole (the provinces of structure and optics being ignored—the former, it is hoped, only temporarily), and accordingly acquire an exclusive importance. The principles of single-circle goniometry are sufficiently well known and the projection-problems incidental to this type of measurement can be systematically dealt with without any further comment (pp. 4—11). But this cannot equally be said of the more highly-developed methods of measurement, which circumstance makes it desirable to include in the present chapter some explanation of general principles—not only of measurement and projection, but also of calculation (pp. 11—19). This notice may, perhaps, prompt a reader to turn immediately to p. 11 and read on to the end of the chapter. Much could be said in support of this procedure, since it amounts to making a general survey of the whole before taking up the detailed study of a part.

I. SINGLE-CIRCLE GONIOMETRY.

The preparation of a projection from single-circle measurements involves various graphical problems. These will be treated in the order in which they usually arise. A concrete example of a typical sequence of operations is given on p. 36.

Problem 1. **To draw the Primitive Circle.**—It is advisable to draw the primitive circle with a *slightly* greater radius (say 5·01 or 5·02 cms.) so that the circle will lie just outside the curved edge of the semicircular part '*C*' of the protractor. The proper radius is, of course, obtained by adjusting the pencil-point compasses on the decimetre

scale '*D*,' or preferably on a good steel ruler (as, *e.g.*, the kind 'No. 279 *D*' manufactured by Chesterman, Sheffield, England). The sheet of paper should be of foolscap size, and should be arranged lengthways from left to right. A sheet of less dimensions should never be used. Occasionally, problems arise involving the drawing of circles with centres lying off the paper; it is then sufficient to slip an auxiliary sheet partly under the first, proper care being taken to avoid any relative movement during the graphical operation.

Problem 2. **To lay off Angles on the Primitive Circle.**—The obvious method is to use the engraved semi-circular edge of the protractor. It is, however, perhaps worth mentioning that the edge '*A*' can be used for the same purpose: for example, in the obtainment of a point *C*, 55° from a given point *D* on the primitive, as shown in Fig. 3. The edge '*A*' is arranged tangentially to the primitive, and with its zero

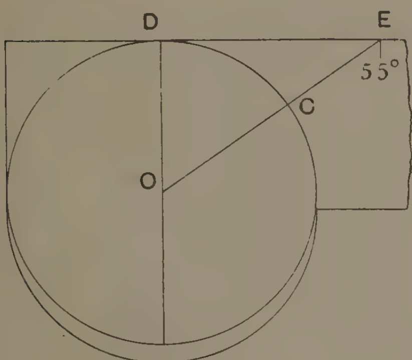


FIG. 3.

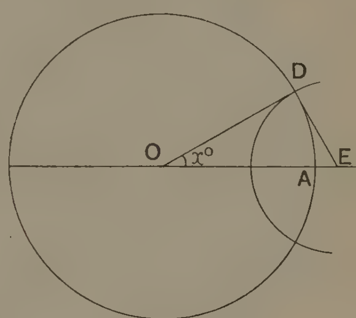


FIG. 4.

at D. The point E is marked at the 55° gnomonic graduation (= 110° stereographic), and the intersection of *EO* with the primitive gives the point *C* required. This alternative method is not recommended, for it is not so direct as the obvious method. It has been mentioned in order to illustrate the truth that the crystallographic gnomono-stereographic scale is identical with that which is placed at the back of the elementary school-boy's ruler for the occasional laying off of such simple angles as 30°, 45°, and 60°. The gnomonic scale is of course one of natural tangents 'multiplied by the radius of the sphere of projection' (or as a school-boy might say 'multiplied by the width of the ruler').

Problem 3. **To draw a Small Circle x° from a given Pole lying on the Primitive.**—This is usually the first serious problem that arises in the preparation of a projection: a certain terminal face *C* is known to make angles $x°$ and $y°$ respectively with two poles *A* and *B* lying in the primitive, and its pole is required. The determination involves

the construction of two small circles which intersect in a pair of points, one of which (generally that which lies inside the primitive) is the point required.

The following simple construction only involves the graduated semi-circular edge C of the protractor (*cf.* Fig. 4). Lay off a point D, $x°$ from A. At D draw a tangent to meet OA produced in E. Then E is the centre and ED the radius of the small circle required.

Now the above construction has, so to speak, been carried out once for all, for all values of $x°$ up to $77°$, and the necessary data engraved on the protractor. It is therefore only necessary to arrange the protractor, so that the centre of the part 'C' corresponds with the centre O of the projection, whilst the near edge 'B' passes through the point A (*cf.* Fig. 5). The points E and D (the latter, of course, being the $x°$ graduation on the semi-circular edge) are marked in (by a fine pencil point or needle), the protractor removed and the small circle (centre E, radius ED) drawn with the compasses.

Another way of effecting the construction has been devised by A. Hutchinson (*Min. Mag.*, 1908, **15**, p. 97, problem 5, Fig. 4; *Zeitsch. Kryst. Min.*, 1909, **46**, 229) in a paper describing the method of using the simplest possible form of protractor, namely a ruler carrying the main gnomono-stereographic scale. In Hutchinson's protractor this scale is repeated on the near edge. In all other exercises described in this book, the zero point of the protractor must always be brought to the centre of the projection, but for this special Hutchinson

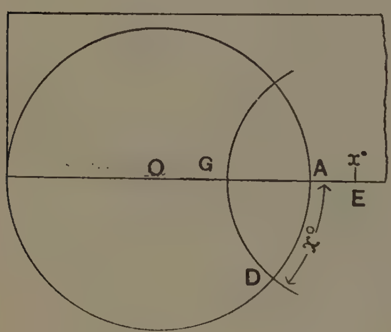


FIG. 5.

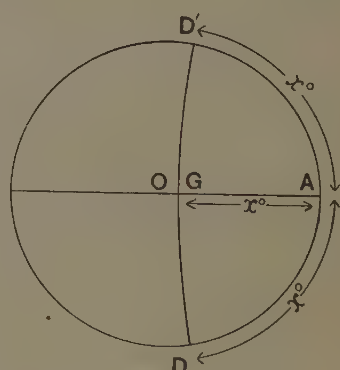


FIG. 6.

method this is not the case. In order to describe the small circle of $x°$ mentioned above, we have to translate the main edge so that the left-hand $(90° - x°)$ graduation occupies the centre; the zero of the scale now locates the point G of Fig. 5 (the 'summit' of the small circle) and the right-hand gnomonic $x°$-graduation simultaneously locates the centre E of the small circle required. It may be added that

the special method of numeration adopted by Dr. Hutchinson facilitates the construction.

Difficulties arise when x has a value greater than 77°. The radius ED is then so great that the 'small circle' cannot be drawn even with the leg extension of the ordinary compasses. In this case some other pole, making angles of less than 77° with the two poles in the primitive, can usually be selected, failing which beam compasses must be used, or better still some form of flexible ruler (first introduced by Wulff; then by Fedorov, Penfield and Herbert Smith). In the latter case the three points necessary to define the circle are obtained as follows (Fig. 6). Two of the points, D and D', are marked off on opposite sides of A, and the third point, G, lying on the radius OA, is located by the main scale of the protractor, for the stereographic distance of G from O is of course $(90° - x°)$. The flexible ruler is then placed symmetrically on the projection, and the arc manipulated till it passes through the points D, G, D'. It may be noted that parts of small circles, corresponding to values of $x = 75°$, 80° and 85°, are included in the part 'C' of the protractor, and can be used as a valuable check on any small circle after it has been drawn. Railway draughtsmen use curve-templets of various values of curvature, and an inexpensive set of flat curves could be produced for crystallographic purposes, provided a demand were guaranteed.

It is of the utmost importance that the initial constructions of small circles be effected carefully, since, generally speaking, all subsequent constructions are influenced by their accuracy. Special care is needed if the observer proposes to pass over to the gnomonic projection from the stereographic.

Problem 4. **To draw a Small Circle x° from a given Pole lying within the Primitive.**—It is convenient (though not obligatory) to draw lightly a diameter through the given point A, along which the main edge must be placed (Fig. 7). The stereographic graduation is noted, and a mental subtraction or addition enables one to mark in the points B and C, $x°$ distant from A. The linear distance BC is measured (best by compass dividers or a millimetre scale). The point D, bisecting BC, is the centre and $BD = DC$ the radius of the small circle. NOTE.—An eventual second small circle may cut the first in two points within the primitive. 'Common sense' must decide which is the intersection required.

Problem 5. **To draw a Zone-Circle through Two Points, One lying on the Primitive.**—In Fig. 8, A is the primitive point, B the other. Then a point C diametral to A, must naturally be a third point on the zone-circle, and the centre of this circle, K, must lie somewhere on ON, the normal to the diameter COA. Further if $x°$ be the inclination

of the zone-circle, it is well known that the angular distance OK is $(90° - x°)$ of the gnomonic scale, or $2\ (90° - x°)$ of the stereographic. Therefore, if $x°$ can be found, the problem is solved.

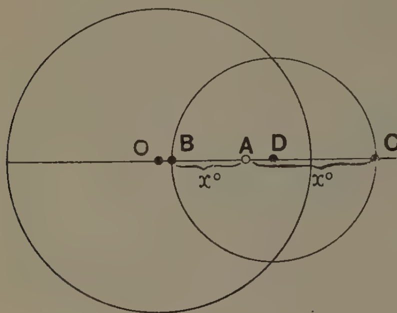


FIG. 7.

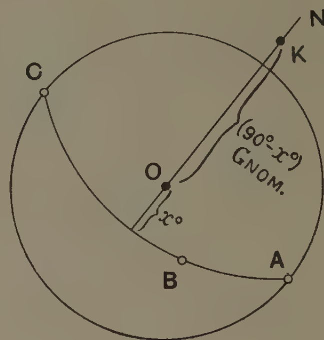


FIG. 8.

To estimate $x°$, place the protractor with the diameter of its semi-circle '*C*' along *COA*. The inclination of the zone *ABC* (though not yet drawn) can be estimated by means of the $5°$-series of the engraved meridians—with an error which does not exceed $\frac{1}{2}°$. Having made the mental subtraction $(90° - x°)$, we now re-arrange the protractor with the main edge along *ON*, and mark in the point *K* from the gnomonic (or stereographic) graduations. With *K* as centre and *KA* as radius describe the circle.

An alternative way, which will become clear after studying the main section on the gnomonic projection, is to rely wholly on the main scale '*A*' of the protractor. We proceed from the stereographic pole *B* to its gnomonic pole, draw a line through the latter parallel to *COA* and measure the azimuth $x°$ gnomonically. Then, as above.

Problem 6. **To draw a Zone-Circle through Two Points Within the Primitive.**—In Fig. 9 the two points are represented by *a* and *b*. The obvious method of solving the problem is to use the part '*C*' of the protractor. This is superposed by trial so that the points *a* and *b* lie proportionately between a pair of the $5°$ meridians. The diametral terminations, *E* and *D*, of the zone to be drawn are marked in, and the inclination of the zone, $x°$, is simultaneously noted. The subsequent procedure is as described above.

An interesting, and sometimes a more accurate, solution is obtained by passing over to the gnomonic projection. By careful alignments of the main edge along the radii *Oa* and *Ob*, the gnomonic poles *A* and *B* are successively marked in. These are now joined and the main edge is then placed along the normal *ON* (*cf.* Fig. 10). The *gnomonic* inclination $x°$, is read, and the point *K*—$(90° - x°)$ gnomonic distance

from O—is the centre, and $Ka = Kb$ the radius of the circle required.

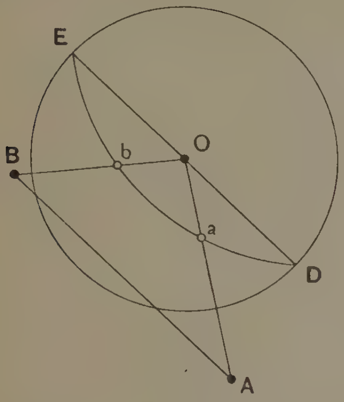


FIG. 9.

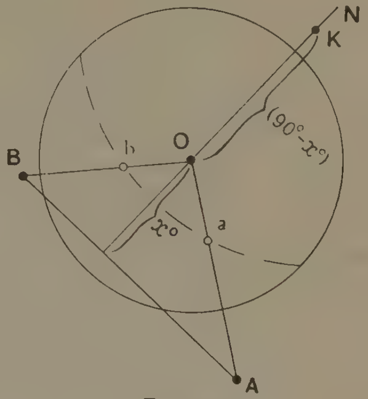


FIG. 10.

NOTE.—Whenever one or both of the poles a and b lie at a greater central distance than 77°, or, conversely, very close to the centre, it is better to rely on the stereographic method.

Problem 7. **To Locate the Pole of a Zone.**—The 'pole of a zone' possesses well known and useful properties (see next problem). If the inclination of a zone is $x°$ (see Fig. 11), then its pole, P, lies on the same radius as the zone-centre K, but at a *stereographic* instead of a gnomonic distance $(90° - x°)$. In other words, the point K is the gnomonic projection of the point P; or, conversely, P lies at half the stereographic distance of K. If the zone-circle has just been drawn, P is naturally derived from K by the main edge of the protractor. If, on the other hand, P has to be located *ab initio*, the inclination $x°$ is determined as in the preceding problem, and the stereographic point, P $(90° - x°)$ distant from O, is marked in.

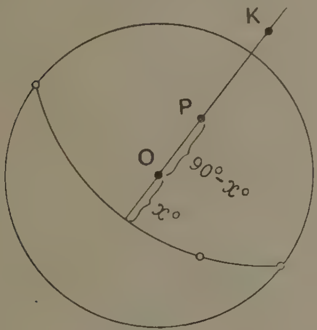


FIG. 11.

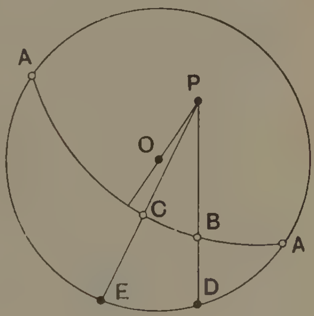


FIG. 12.

Problem 8. **To Locate the Position of a Face-Pole from an Interfacial Angle.**—As is well known, this is effected by laying off the angle

on the primitive and projecting it to the zone-pole P (*c.f.* Fig. 12). Thus a face-pole B, lying in a zone AB and $x°$ distant from A, is located by laying off $AD = x°$, on that part of the primitive which lies on the convex side of the zone, and projecting D to P. Any other point C (of known angular distance from A or B) is similarly located, an angle BC being of course marked off from D instead of A.

Problem 9. **To Determine the Value of an Interfacial Angle in Projection.**—This problem, the converse of the preceding, is simply solved by the help of the semi-circular part '*C*' of the protractor (*cf.* Fig. 13). The latter is superposed with its diameter on AA' and the

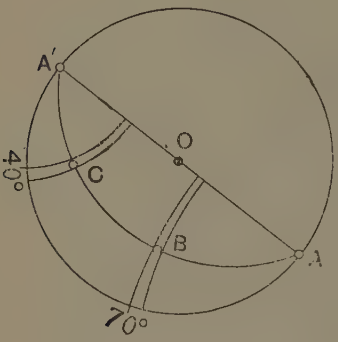


FIG. 13

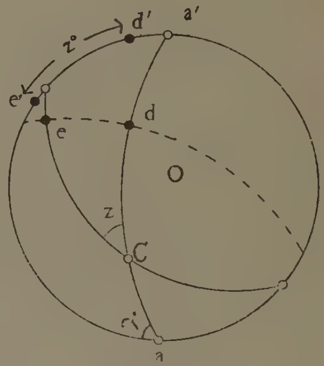


FIG. 14.

angles AB (say 68°) and $A'C$ (say $42\frac{1}{2}°$) are estimated from the engraved small circles. Then $BC = (180° - 42\frac{1}{2}° - 68°) = 69\frac{1}{2}°$. It is believed that the following note deserves the careful consideration of all single-circle workers.

It cannot be too strongly urged that this method of estimating zonal angles satisfies all requirements in the determination of the system of crystallisation, and can advantageously replace the measurement of 'superfluous' interfacial angles on the goniometer. The most remarkable feature of crystal development is the way in which a relatively large number of faces are included in a small number of zones; as a result, the purely geometrical relations can be established by a small number of zone-measurements (generally not more than five in the anorthic system). It is, however, also necessary to determine the system, and this has generally been effected in the past by the measurement of additional zones and a subsequent comparison of the various angular values. Now the necessity to measure additional zones really disappeared with the introduction by Fedorov of transparent stereographic nets, and by Penfield of a protractor of the type reproduced as '*C*' in the present instrument. If sufficient measurements have been carried out for the preparation of a projection, the additional measurements necessary for the correct diagnosis of the system, can be effected graphically with the necessary degree of accuracy. This means an immense saving of labour, for the measurement of additional zones is almost inevitably repeated on all the crystals measured, the result being an immense number of angles, which it is customary to check by logarithmic calculations. This additional double labour (measurement and calculation) has no compensating advantage, for superfluous angles can never have any theoretical significance, nor would they seem to have any practical value; they are, for example, of no help in the identification of a crystal by Fedorov's method of crystallochemical analysis.

Problem 10. **To Measure the Angle between Two Zones.**—In the special and most frequent case where one of the zones is the primitive (the angle a, for example, of Fig. 14), the problem is solved by arranging the semi-circular part '*C*' with its diameter on the diameter aa', and estimating the angle from the relative inclination of the zone between the nearest engraved meridians. With ordinary care the error does not exceed $\frac{1}{2}°$.

In the general case where both zones Cd and Ce, are inclined, the following procedure is adopted. The main edge of the protractor is aligned with OC, and the angular distance OC is noted. The point C is then regarded as the pole of an imaginary zone, which is drawn by first locating its centre, K (not shown in the Figure) in the usual way. This zone cuts the two original zones in e and d. The angle ed is an accurate measure of the angle eCd, and can be estimated either by projection from the pole C to the points e', d' of the primitive or by the direct application of the small circles engraved on the protractor.

Note.—The above construction depends on the truth that an angle '$z°$' between two zones Ce and Cd is accurately measured by the meridional circle passing through two points e and d, 90° distant from C. A much simpler method, however, depends on the fact that the stereographic projection preserves angular truth, the angle required being that between tangents to the two circles at the point C. Now this angle is in turn measurable by an angle KCK', where K and K' are the centres of the two zone-circles (not shown in Fig. 14) and, provided these centres are not beyond the 154°-graduation of the protractor, this is the most direct method of solution.

Problem 11. **To Draw a Zone z° inclined to a Second Zone.**—Let Cd be the given zone (Fig. 14) and C the point through which the second zone Ce has to be drawn. The imaginary 'zone of which C is the pole' (or more briefly expressed the 'Pole of C') is drawn and the point d, thereby located, is projected to d'. An angle $d'e' = z°$ is marked off on the primitive and the projection of e' to C locates the point e. The zone required is that which passes through C and e. This and similar problems admit of elegant treatment by the gnomonic projection.

II. TWO- AND THREE-CIRCLE GONIOMETRY.

1. **Two-Circle Goniometry.**—In sharp contrast with the preceding single-circle goniometry, the preparation of a projection now becomes a simple, uniform process, and therefore one which can be carried out with enhanced speed and accuracy.

This improvement, as will be seen presently, is really due to the fact that the two-circle instrument attacks what is obviously a three-dimensional problem in a three-dimensional way, and not in a disconnected series of two-dimensional stages. One initial adjustment on the two-circle goniometer (entailing no more work than the average single-circle adjustment) admits of the complete geometrical determination of the singly-terminated crystal, so frequently met with in the

province of mineralogy; a corresponding second adjustment being naturally required in the case of the doubly-terminated crystal, more typical of laboratory products. That the method of measurement is perfectly general, being equally applicable to a crystal obeying the zone-law (or the law of simple multiple intercepts); a crystal exhibiting both plane and curved faces; a cut gem or any other artificially or naturally fashioned polyhedron, will presently become obvious, but will be left out of further consideration since we are here mainly concerned with the relatively simple, plane-faced development of the average crystal.

2. **The Zone Adjustment.**—There are two distinct methods of effecting the preliminary adjustment of the crystal on the goniometer.[4] The first, which may be suitably termed the 'zone-adjustment,' is illustrated in projection by Fig. 15, representing a few of the forms

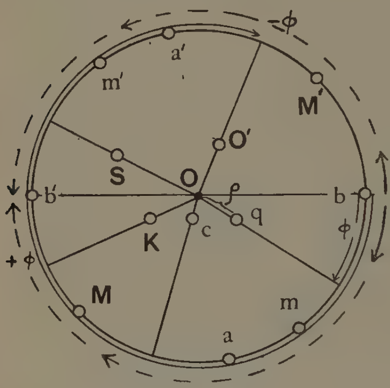


FIG. 15.

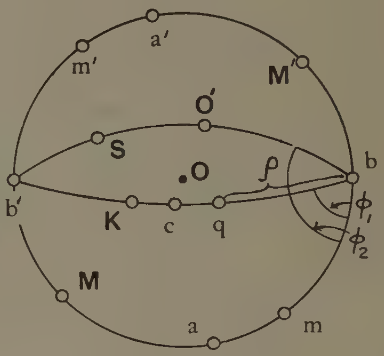


FIG. 16.

developed on copper sulphate (more fully described in Chap. IV). The crystal is here adjusted by means of the vertical zone, in exactly the same way as on a single-circle goniometer. The interfacial angles in this zone are then read from one of the circles in the usual way, but the presence of the other circle (aided of course by the general construction of the instrument) now permits us to take readings of a peculiar kind for all the terminal faces of the crystal (*i.e.* those not adjusted in the sense of single-circle goniometry). These latter read-

[4] There is really no absolute necessity to adjust the crystal, for if the latter be placed in any random orientation the two circles admit of a precise geometrical investigation. It is interesting to note that measurement without adjustment was the method adopted by W. H. Miller in the two-circle instrument, which he rigged up (and subsequently dismantled) for the investigation of a facetted bead of platinum exhibiting no less than 147 faces (*cf.* the posthumous paper, edited by W. J. Lewis, *Proc. Camb. Phil. Soc.*, 1882, **4**, 236). In Miller's few notes, occupying less than a page of print, the general principles of two-circle measurement and projection are vividly outlined. The primary object of adjusting a crystal on the two-circle goniometer is to make possible a comparison of two or more crystals. It goes without saying that the calculations are also simplified thereby.

ings, generally designated ϕ ('phi') and ρ ('rho')—where ϕ refers to the vertical, and ρ to the horizontal circle of the Fedorov-Goldschmidt type of instrument—are neither the interfacial angles of single-circle goniometry nor the interzonal angles of single-circle calculations. Their nature is indicated in Fig. 15. In the typical case of the face q the angle ϕ is seen to represent the angle between the radius Ob and the radius Oq on which the face-pole is located, whilst ρ is the angular distance from the North pole of the sphere of projection (in other words the angle of slope of the face q on the crystal); so that ϕ and ρ respectively represent the longitude and co-latitude (or polar distance) of the face-pole on the sphere of projection. The same is true of all other terminal faces, the angle ϕ being reckoned clockwise from the radius Ob up to the value of $360° = 0°$, whilst ρ is reckoned from the centre, whereby the vertical faces naturally acquire an identical ρ-value of $90°$. NOTE.—Some authors (including V. Goldschmidt) prefer to count the angle ϕ in a counter-clockwise manner (shown in Fig. 15 by the interrupted peripheral arc) in the special case of any face-pole lying behind a zero diameter $b'Ob$, and prefix (or superscribe) a minus sign. Thus the ϕ-value of the face-pole O' *becomes*—$67°39'$, instead of $+292°21'$.

With regard to the preparation of a stereographic projection, it will be realised that this problem is of a most elementary description. The ϕ-reading of our typical face q is marked on the primitive; the main edge A of the crystallographic protractor placed along the radius Oq, and q marked in from the ρ-reading. The process is obviously effected both speedily and accurately. The same may be said of the actual measurement, for the uniform method of bringing each face into the reflecting position makes for mechanical speed in spite of the fact that the measurement does not proceed on zonal lines.

3. **The Face Adjustment.**—This type of two-circle adjustment is illustrated by Fig. 16. The crystal is adjusted so that a well developed face, say $b(010)$, is parallel to the plane of the vertical circle of the Fedorov-Goldschmidt type of instrument—a form of adjustment which cannot be effected on a single-circle goniometer. Subsequent movements of both circles now allow the measurement of all zones which pass through this face-pole $b(010)$, and the readings acquire a special interest. In the typical case of q, the angle ϕ_1 now represents the *interzonal* angle abq (the vertical zone ab being conveniently selected as the zone of reckoning—the 'zone of reference'), whilst ρ represents the *interfacial* angle bq. This is true for all the readings without exception, so that the system of measurement is to be regarded as a remarkable extension of single-circle work, interfacial and interzonal angles being measurable without distinction.

With regard to projection, the process is uniform throughout but not quite so easy as in the previous case described. Zones have now to be drawn through $b(010)$ in accordance with the interzonal ϕ-readings, and the various face-poles marked in from the interfacial ρ-readings. The latter operation involves the location of the zone pole, as in single-circle goniometry. The plotting of these zone-poles is carried out simultaneously with the fixing of their geometrical centres (at twice the central angular distances) by simply laying the main scale A of the protractor along the diameter perpendicular to $b'Ob$.[5]

4. **Stereographic Nets.**—These have long been employed in the sciences of astronomy and cartography and to some extent in navigation. Apparently all three possible types were first applied to crystallographic purposes by Fedorov (including one that is widely known as the 'Wulff net') so an impersonal form of nomenclature seems to be

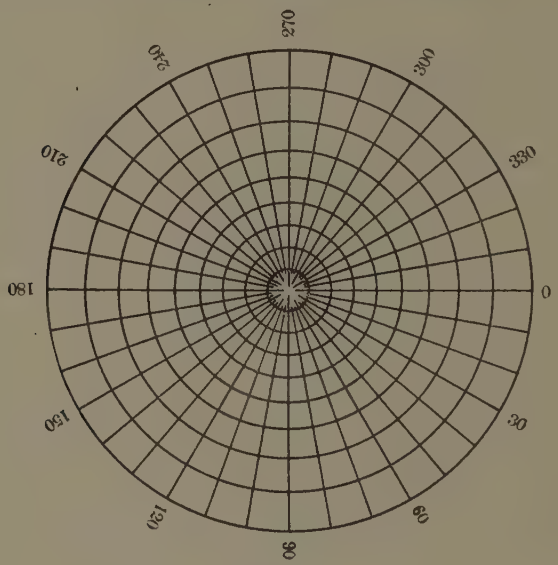


FIG. 17.

called for. If meridians of longitude and parallels (small circles) of latitude be inscribed on a sphere, the view from the North pole is that of Fig. 17; and from a point in the Equator, that of Fig. 18. They may therefore, although not very satisfactorily, be termed Polar and

[5] It must be pointed out that there are cases in which the 'zone-adjustment' becomes identical with the 'face-adjustment.' Suppose the crystal of Fig. 15, adjusted by the vertical zone, to be orthorhombic and not anorthic, then the face $c(001)$ lies at the centre of the projection, and any rho-angle is the angle which a face makes with this face $c(001)$—no matter whether it is developed on the crystal or not. Similarly the phi-angle for any face (hkl) becomes equal to a potential angle $010 : hk0$ and may alternatively be regarded as an interfacial angle or an interzonal angle. The above relations of course hold in every case in which the axis of the zone adjusted is an axis of symmetry.

Equatorial. A combination of the first with a pair of the second kind—one with meridians radiating from b(010), the other from a(100) —is the *generalised* net which plays such an important rôle in the provinces of optics and crystallochemical analysis. Figs. 17-18 are merely general patterns; in actual nets the meridians and small circles may be drawn every 2° or 5° instead of every 10°, the number that can be conveniently included without confusion depending on the scale, type and colour of ink. So far as their limited application to single-circle work is concerned they are best printed either on transparent paper (Fedorov) or on a stout cardboard (Wulff), their office being to guide drawing operations on a second sheet of paper; but for the purposes of two- and three-circle goniometry they are best printed in colour (*not* black) on a stoutish paper. The various poles can then be plotted in pencil or ink and show up well against the faint tint of the curves, any eventual graphical treatment being thereby

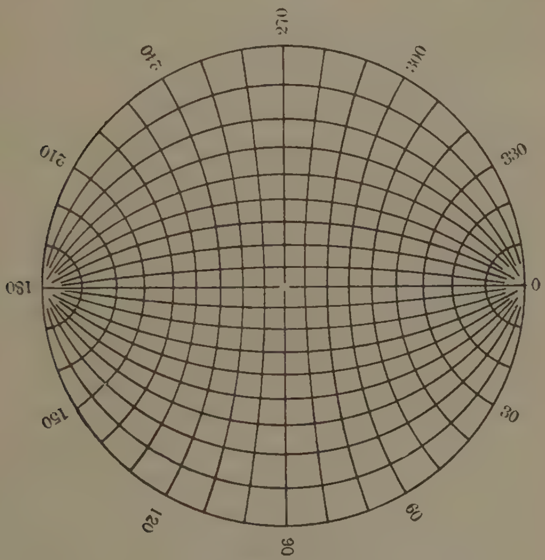


FIG. 18.

facilitated. A comparison of Figs. 17-18 with Figs. 15-16, respectively illustrating the zone- and face-adjustments, will show that two-circle readings can be plotted directly on the appropriate form of net, but a protractor remains indispensable in any subsequent graphical treatment.

5. **The Efficiency of Two-Circle Goniometry.**—This can only be considered here from the standpoint of measurement. If a crystal were perfect (*i.e.* obeyed the zone-law or law of simple multiple intercepts *rigorously*) then two-circle goniometry would imply a certain amount

of superfluous measurement. This is, perhaps, best illustrated in terms of Fig. 16, from which it will be realised that the lie of the zone *bqcK* is determined by a single ϕ-reading, say that of q; consequently it would be superfluous in a perfect crystal to take corresponding ϕ-readings for the faces c and K. But every practical crystallographer knows that a crystal is not perfect. The faces *bqcK* do not rigorously lie in a zone, and successive ϕ-readings (as well as ρ-readings of course) are necessary to fix the positions of c and K *as they are actually developed on the crystal.* From the most exact point of view there is, then, not a single superfluous measurement in the province of two-circle goniometry—either with face-adjustment or with zone-adjustment. The face adjustment has, however, a slight advantage, since these small deviations from a rigorous law can be mutually adjusted at an earlier stage. The ϕ-readings for q, c and K, for example, can be averaged to a single value, even before the projection is made.

The above property of two-circle goniometry, reinforced by the additional circumstance that only one or, at the most, two adjustments (the latter, in the case of a doubly terminated crystal) are necessary for a complete measurement, leads to a great saving of time and labour. A series of comparative trials leads the author to the conclusion that the relative efficiencies (as measured by the quotient, results/time) of single and two-circle instruments are 4 : 7; *i.e.* as much work can be effected by the two-circle instrument in four hours, as in seven hours devoted to single-circle goniometry. Moreover, the readings are necessarily more faithful to the crystal.

6. **The Graphical Determination of the System.**—The view has been expressed on a previous page (p. 10) that the system should as far as possible be determined graphically, even in single-circle goniometry. This procedure is always adopted in two-circle work. In some nine cases out of ten the system of crystallisation is obvious from a glance at the projection (which means that the habit of a crystal is generally in close accord with the symmetry). The tenth case, however, may be deceptive: even in projection, a crystal may appear to be anorthic, when it is really monoclinic; monoclinic when it is orthorhombic or rhombohedral; orthorhombic when tetragonal; and, apparently, in one case in a life-time, anorthic when tetragonal—in each case owing to a distortion which leads the observer to mount the first crystal examined (of our supposedly new substance) on the gonimeter in the wrong way (*i.e.* in a way not in accordance with the symmetry. An analysis of this first crystal must therefore always be made in projection, and the result confirmed, whenever possible, by a simple optical examination. The apparently more difficult case of Fig. 15, in which only one zone is

actually measured, need only be considered. All other zones can be readily recognised, as indicated by the interrupted lines of Fig. 19, by viewing the projection through the part C of the protractor (suitably orientated). The interfacial angular values are simultaneously noted

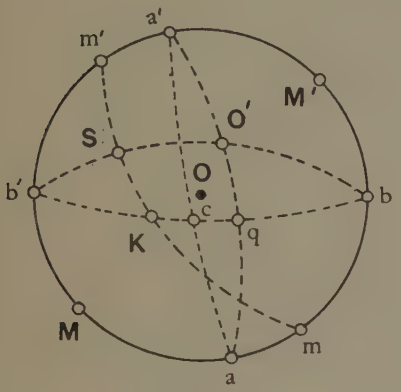


FIG. 19.

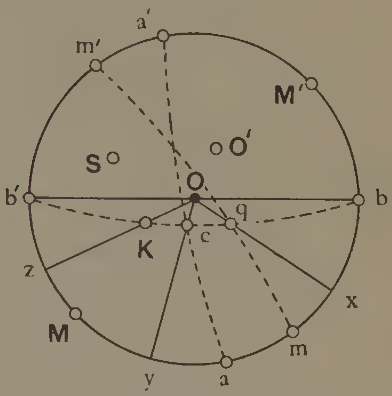


FIG. 20.

by the help of the engraved small circles (with an error rarely exceeding $\frac{1}{2}°$). A subsequent comparison of these graphically determined angles, both with each other and with the measured angles of the vertical zone, reveals any symmetry which hitherto may have escaped detection.

7. **Two-Circle Calculations.**—Although formally lying outside the limits of this chapter two-circle calculations may, perhaps, now be usefully outlined quite generally in view of the special form of treatment subsequently adopted in Chapter VI. Since this is not a monograph on goniometry it is not proposed to deal with forms of calculation peculiarly adapted to two-circle work, but to outline the principle of such forms as would come most naturally to a single-circle worker who is proceeding to two-circle work.

A fully worked-out example is given by C. M. Viola (*Grundzüge der Kristallographie*, 1904, p. 90), for the zone-adjustment variant. The principle is quite simple. In the solution of an oblique spherical triangle (except in the special cases where three sides or three angles are given) it is a well-known computer's device to resolve it into a pair of right-angled components by imagining a perpendicular arc to be drawn from an angle to the opposite side (produced if necessary). These two components are then solved piecemeal by the simple formulæ of right-angled spherical trigonometry. Now this type of resolution is actually effected, quite mechanically, by the two-circle instrument, as will be seen from a simple consideration of the relation of the face q (with its ϕ-reading bx and its ρ-reading $Oq = 90° - qx$)

to the oblique spherical triangle *mbq* (as shown by Fig. 20). Assumed we take *qx* and *bx*, as two of the necessary five fundamentals, then the angle *qbx* (the supplement of β), as also the interfacial angle *bq* if it be wanted, follows immediately. It is on such simple rectangular principles that the theoretical value of an angle (no matter whether of the ϕ or ρ or interfacial or interzonal type) may be computed. The elements follow on the usual lines. NOTE.—For an interesting relationship between the ϕ-values of four tautozonal faces (the angles *bx*, *by*, *bz* in the specific case of *b*, *q*, *c*, *K*) and the corresponding indices, see p. 65. In this particular case the formula runs: $\cot bx + \cot bz = 2 \cot by$.

The alternative form of adjustment, that represented by Fig. 16, needs no special description since the angles measured are wholly interfacial or interzonal. There is, however, one novel feature that deserves emphasis. The interzonal angles *abq* (the supplement of β) and *abO'* have been measured and may be adopted (amongst other angles) as fundamentals. In which case it is possible to compute an axial ratio *c* : *a* from the corresponding ratio of sines, a preliminary solution of one or more oblique spherical triangles being dispensed with.

8. **Three-Circle Goniometry.**—The addition of a third circle would appear to bring no such fundamental extension of principle in crystal measurement as that involved in the addition of the second circle, but it allows a more flexible handling of the crystal. In two-circle measurement it sometimes happens that a symmetrical crystal has been adjusted with a non-symmetrical orientation, and a new adjustment becomes desirable for purposes of publication. With the everyday crystal this is just as easily effected as in the case of single-circle goniometry; but if the crystal be excessively *minute*, there is a possibility or probability (depending on the size) that the desired orientation cannot be effected. It is in such extraordinarily difficult cases that the three-circle goniometer can be depended upon to answer every requirement, for the third circle makes it possible to measure a series of interzonal and interfacial angles, not merely from one face, but from any number of faces belonging to the zone initially adjusted. Preliminary explorations are thereby possible, enabling the observer to make the most suitable choice of a base of measurement. If the crystal be doubly terminated, it has of course to be re-adjusted as in two-circle work in order to get the full set of readings from both ends of the crystal. A further possible application of the three-circle goniometer is mentioned on page 67. It may also be applied to the mechanical solution of oblique spherical triangles. Perhaps the best form of three-circle goniometer is that devised by G. F. H. Smith (*Min. Mag.* 1899, **12**, 175; *Zeitsch. Kryst. Min.* 1900, **32**, 209). So far as the

preparation of a stereographic projection is concerned, the problem is that of the face-adjustment type of two-circle goniometry which has already been considered.

EXERCISES.

General Note.—Some of the following stereographic projections will be subsequently required as material for exercises on the gnomonic projection, and should accordingly be preserved. Projections should never be made on small scraps of paper. The radius of the primitive circle should always be 5 cms. (or 7 cms. in the case of the Penfield apparatus). The lower hemisphere of projection should be omitted in every case.

Solutions or answers to most of the exercises will be found on p. 113, *sqq*. With regard to the numerical exercises given in later chapters, it may be explained that they are all actual (not invented) examples. Many have been borrowed from V. Goldschmidt's *Winkeltabellen*, and, in view of the accuracy of this lexicon, it has not been thought necessary to work out the figures independently. As a result it will occasionally happen that the value obtained by a reader differs from the 'answer' given at the end of the book by $\frac{1}{2}'$, $1'$ (or even $2'$?)—for Goldschmidt's angular values are derived from the elements and not from each other as in most of these exercises. If a worked-out value differs from the 'answer' by more than $2'$, it can be taken that there is an error in the working, or alternatively, a misprint. Any intimation of misprints or author's errors will be gratefully received. Perhaps no student should attempt to work out all the numerical examples belonging to a chapter before proceeding to the next. Possibly a better plan *in general* would be to solve a representative selection under each section so as to acquire a general working knowledge of the various devices as a whole, for the proper field of application is independent investigation. When a more difficult problem comes up for solution (say, once a year in the case of an active researcher) its prototype will generally be found towards the end of each set of exercises. Such relatively difficult problems have been asterisked. It need scarcely be added that (in any graphical work) angular values, measured to minutes, should be rounded off instantly to $1/2°$, $1/3°$, $1/4°$ as the case may be (or to $1/6°$ or so towards the end of the gnomono-stereographic scale). Any anxious attempt to plot, say, $36°34'$ might easily lead to a greater error than a plotting of $36\frac{1}{2}°$ or $36\frac{2}{3}°$. Speed with accuracy comes with practice. Better slow and sure, to begin with. A final remark: a single-circle worker should not pass by exercises in which two-circle data are quoted. Their function is to save his time in the mechanical plotting of stereographic poles

and so allow him to come to grips with the real problem more rapidly. All two-circle data are to be interpreted in the sense of Fig. 15 (p. 12), this being the more convenient form for plotting purposes.

Exercise 1.—Prepare a general stereographic projection of the forms *c* (001), *a* (100), *m* (110), *e* (101), and *p* (111) of the tetragonal idocrase; given the angle *ce* = 28°15′. Locate the pole of the zone (101) : (010) and mark in the face poles (131) and (121) from the angles 010 : 131 = 35°9′, 010 : 121 = 46°34′. Read the inclination of the zone [100 : 121] through the part *C* of the protractor, say *x*°. With the main edge locate the centre of the zone circle, *twice* (90° — *x*°) ster., and draw it. Locate its pole at (90° — *x*°) ster. and plot the position of (221) from the value (100) : (221) = 53°48′. Finally plot the positions of two faces *x* and *y* from the following data, *x* : (100) = 65° 37′, *x* : (010) = 34° 20′, *y* : (100) = 73°46′, *y* : (010) = 56°0′.

Exercise 2.—Prepare a general stereographic projection of the forms *c* (001), *b* (010), *m* (110), *d* (101), *f* (011), and *o* (111) of the orthorhombic topaz, given the angles *bm* = 62°9′, *co* = 63°54′. Also plot the following poles: *y* lying in the zone *cb* with *cy* = 62°20′; *i* and *u* lying in the zone *cm*, with *ci* = 34° 14′, *cu* = 45° 35′; *v* where *av* = 69° 51′ [*a* being, of course, (100)], *bv* = 56°54′; *w* where *aw* = 53°22′, *bw* = 50°54′.

Exercise 3.—Prepare a general *stereographic* projection of the anorthic copper sulphate from the data and by the method outlined on p. 36. If the various constructions go haltingly, repeat the exercise the next day, and a third time after a week. In the meantime, proceed to the simpler exercises of Chapter III on the *gnomonic* projection.

CHAPTER III.

THE GNOMONIC PROJECTION AND ITS RELATION TO THE STEREOGRAPHIC.

'Of these [projections] the stereographic offers many advantages on account of the facility and the accuracy with which the distances between the originals of any two points may be measured; or the points determined in the projection, having given the mutual inclinations of the faces they represent. In the gnomonic projection, the corresponding projections are less simple. The projection labours also under the disadvantage that the half of a crystal cannot, as in the stereographic projection, be exhibited on a single surface of finite extent. On the other hand, great circles being projected into straight lines, the zones to which a given face belongs can be very readily ascertained; and the situation of a face common to two zones can be much more easily determined than in the stereographic projection. There are also constructions of great simplicity for finding the symbols of points in the projection, or for laying down the points when their symbols are given, depending upon the equality of the anharmonic ratios of points and great circles on the sphere, with those of the points and straight lines into which they are projected gnomonically, which I now proceed to investigate.' W. H. MILLER (1859).

'Parmi les problèmes auxquels donne lieu le calcul des cristaux, un grand nombre, et des plus intéressants, se trouvent donc résolus très-simplement par la projection gnomonique. La projection stéréographique a, il est vrai, pour avantages, de se contenter d'un cadre plus restreint, et de ne pas donner de points situés à l'infini; mais elle n'est jamais qu'une image destinée à représenter la position relative des pôles. La projection gnomonique est une véritable épure géométrique qui permet de trouver graphiquement, et souvent à simple vue, la solution de beaucoup de questions importantes.' E. MALLARD (1879).

'Die stereographische Projection steht an Bedeutung für krystallographische Arbeiten hinter der gnomonischen zurück. Sie ist weniger geeignet zur graphischen Behandlung von Aufgaben. Dagegen ist sie gut zur Vermittelung mancher Anschauungen. Jedenfalls ist sie dem Krystallographen ein wichtiges Hülfsmittel, und es ist für ihn nöthig, sich soweit mit ihr bekannt zu machen, dass er stereographische Projectionsbilder rasch und sicher ausführen kann.' V. GOLDSCHMIDT (1899).

1. **The Gnomonic Projection.***—Unlike the stereographic, the gnomonic projection does not involve a preliminary projection on a sphere. Face-normals are merely drawn from the 'centre of the crystal' till they meet a plane, the points so obtained being the gnomonic poles. This process is illustrated for a simple tetragonal crystal (idocrase = vesuvianite) by Fig. 21, in which a horizontal plane (normal to the vertical axis of the crystal) has obviously been selected as the plane of projection. The projection of *c* (001) is seen to centre the parallelogram carrying the poles of the upper pyramidal faces (it will be seen presently that this 'parallelogram' is really a square when viewed in plan). Further, it is clear that the prism-normals will never meet the plane of projection; nor will the normals of any lower faces, unless they be replaced by their parallel faces. It will already

* The term 'gnomonic' is derived from the Greek '*gnomon*,' which has various significations, including gnomon (in Euclid's sense); the index of a sun-dial; and the carpenter's square (Lat. '*norma*').

be seen that zones are projected as straight lines, but this is by no means the only advantage possessed by the gnomonic projection.

2. **Its Relation to the Stereographic.**—This admits of easy explanation in terms of Fig. 22. The crystal is supposed to be arranged with its centre at O. Then a face-normal OV, meeting the sphere in V, is projected stereographically to the point T of the equatorial plane, where $OT = r \tan OST = r . \tan (NOV/2)$. Now suppose the face-normal, OV, be prolonged till it meets the upper plane (tangent to the sphere at N) in the gnomonic pole G. It is clear that NG is parallel to OT, and that $NG = r \tan NOV$—instead of $r . \tan (NOV/2)$

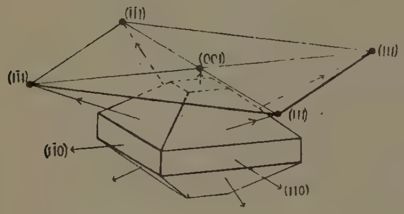


FIG. 21.

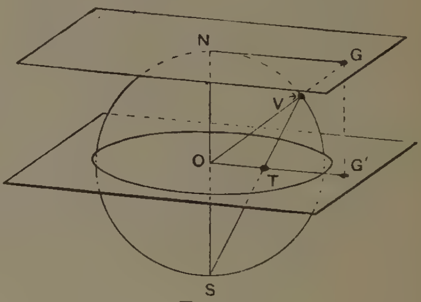


FIG. 22.

as in the stereographic. Now suppose we imagine the crystal to remain centred at O for stereographic purposes, but lowered through a vertical distance r to the point S for gnomonic purposes, the upper (gnomonic) plane being correspondingly lowered through the same vertical distance $r = NO$. We have now got the two projections on the same plane; G has fallen to G', lying on OT produced; $OG' = NG = r \tan NOG$ (or NOV) $= r . \tan (2\, NOV/2)$. Therefore **the gnomonic pole of any face, in a combined gnomono-stereographic projection, lies on the same radius as the stereographic pole, but at twice the central angular distance.** Accordingly it is the simplest possible matter to pass from one projection to the other. Suppose, for example, we have prepared a stereographic projection of the upper half of the crystal of idocrase (the lower half can be neglected for all purposes except that of elementary teaching), and it is desired to proceed to the gnomonic. The main scale '*A*' is aligned with a radius carrying a stereographic pole (denoted by a tiny circle in Fig. 23); the stereographic distance is noted, and the corresponding gnomonic pole is marked in (as a point-circle in the Figure) at twice the stereographic graduation. The stereographic poles of the vertical faces, lying at angular distances of 90° from the centre, must be left alone as the protractor cannot be carried to 180° (infinity) and is suitably terminated at 154°. The gnomonic poles of the upper pyramidal faces are seen to map out a square, with the gnomonic pole of the face c at its centre. The zones

are projected as straight lines, which obey a simple linear law—of fundamental, crystallographic importance, since it enables one to determine the indices of any gnomonic pole by an inspection of its position in the diagram.

3. **Neumann's Gnomonic Theorem.**—In Fig. 24 the stereographic poles of the (111) faces have been omitted, and the gnomonic poles of (101)—derived from their corresponding stereographic poles by laying off twice the central angular distance—have been inserted, and are seen to lie midway between adjacent (111) poles. It is seen that the diagram falls naturally into four component squares. Now a plane diagram of this regular pattern immediately invites the application of the methods of co-ordinate geometry. If we take the gnomonic pole

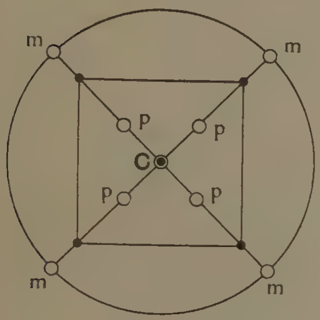


FIG. 23.

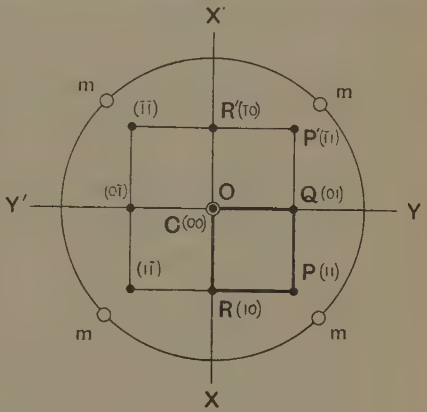


FIG. 24.

of c (which happens to coincide with the centre O) as origin, and XOX', YOY' as axes, and the lengths OR, OQ as unit lengths along these axes respectively (they happen to be equal in this case, since the crystal is tetragonal), we can specify the positions of the gnomonic poles in terms of these lengths. Thus, the co-ordinates of c (001) are (00); similarly we have R (10), P (11), Q (01), P' ($\bar{1}$1), R' ($\bar{1}$0), and so on. *Now if we add* 1 *as third digit, we immediately get the Millerian Indices.* This is true for all gnomonic poles (since 'gnomonic poles' for the vertical faces can scarcely be said to have a real existence).

The profound importance of this truth can now be illustrated. Suppose a crystal of idocrase exhibits a face i. The first task of the single circle worker is to fix its geometrical position by suitable measurements, say, by measuring the angle which it makes with the two faces a (100) and c (001). From these two measurements he will establish its position in the stereographic projection (see Fig. 25). The next task is the determination of indices. This is most simply effected by proceeding to the gnomonic pole I, by means of the main

edge '*A*' of the protractor. The co-ordinates of *I* are seen to be (3/2, 1/2). Adding 1 as third digit, we get 3/2, 1/2, 1; and clearing up fractions, by multiplying throughout by 2, we get *i* (312). The same method is also illustrated in Fig. 25 for a face *l*. What are its indices? Obviously (243): since (2/3, 4/3) ⤚ (2/3, 4/3, 1) ⤚ (243). There is, in general, no necessity to derive zone-symbols (and cross multiply them), or to resort to formulae of computation, except in the rare cases of highly complicated indices: almost always to be regarded either as vicinal faces (*c.f.* H. A. Miers, *Phil. Trans.* [*A*], 1903, **202**, 459), or as corrosion surfaces in contradistinction to plane faces of growth.

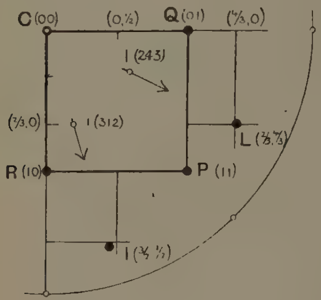


FIG. 25.

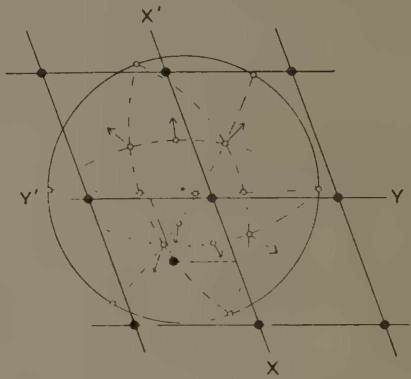


FIG. 26.

In the tetragonal case described the 'primitive gnomonogram' (00) — (10) — (11) — ($0\bar{1}$) is a square. In the cubic system it remains a square, but with a side equal to the radius of projection (*cf.* Fig. 27). In the anorthic system it becomes a parallelogram with the corner (00) displaced from the centre *O* of the primitive, as seen in Fig. 26 representing the α-modification of methyltriphenylpyrrolone—a rare organic compound described by A. E. H. Tutton (Groth's *Chemische Krystallographie*, **5**, 535). This example is possibly the only instance in the crystal kingdom, which without a change of indices fulfills all the demands of a convincing demonstration. As in the case of idocrase (Fig. 24), we have, as terminal faces, the basal plane (001), four (101)-faces and four (111)-faces, so that the quadrupled primitive gnomonogram is complete. The isolated terminal face lying at the centre of one of the parallelograms has the co-ordinates ($\frac{1}{2}$, $-\frac{1}{2}$), and the indices are therefore ($\frac{1}{2}$, $-\frac{1}{2}$, 1), *i.e.* ($1\bar{1}2$). The vertical faces, which can be only shown stereographically, are (010), (100) and ($1\bar{1}0$). The gnomonic poles, as always, are obtained from the stereographic by marking off twice the angular distance from the centre by means of the crystallographic protractor as indicated in Fig. 26 by the various arrows.

The Neumann method is applicable *in all its simplicity* **to all systems provided one of the crystallographic axes, OX, OY, OZ is projected vertical.** In view of the various orientations generally adopted in crystallography, this means that it is only the rhombohedral system (as distinguished from the hexagonal) that requires a special treatment. In the case of the hexagonal system one of the first three axes has to be provisionally discarded, the corresponding index being put in finally by inspection. A discussion of these two special systems will be found on p. 76.

It is important to note that a crystal should never be projected on the plane of a face (unless the face happens to be perpendicular to a zone-axis). Thus, an anorthic crystal should never be projected in such a way that the pole of the face *c*(001) occupies the centre of the projection, for when so projected there is no longer any vertical zone (actual or potential) and the gnomonic diagram ceases to be a reduplicated parallelogram; and although it has been shown by C. Travis (*Zeitsch. Kryst. Min.* 1910, **47**, 586) that this system of converging and diverging lines is amenable to graphical treatment by the help of a special protractor, there can scarcely be any doubt that it is better to avoid distortion at the outset by arranging the projection with a zone vertical. It is for this reason that the measurement of an anorthic crystal with face-adjustment is always followed up by a projection in which the pole of the adjusted face is brought into the stereographic primitive instead of the centre of the projection (*cf.* Fig. 16, p. 12—whether this face is labelled *b* or *c* or any other letter is, of course, a matter of convention).

It will, perhaps, be clear from the above that the gnomonic projection is quite accessible to the single-circle worker. It must, however, be emphasised that reasonable care must be taken in the initial preparation of the stereographic projection, if the gnomonic is to fulfil one of its most important functions—the determination of indices. Fig. 25 illustrates the degree of accuracy ordinarily attained by the use of the crystallographic protractor; the gnomonic poles of *i* (312) and *l* (243) are seen to lie very close to their theoretical positions. Vertical faces naturally demand special treatment (see p. 37).

A reader who wishes to become practically acquainted with the gnomonic projection could scarcely do better than prepare simple stereographic projections of all the systems,

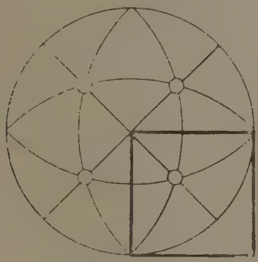


Fig. 27.

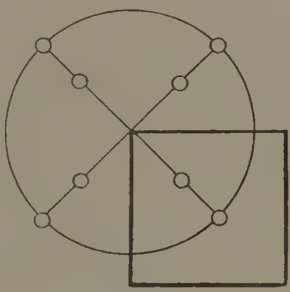


Fig. 28.

and proceed to the gnomonic, by laying off twice the central angular distance of each pole. He will then find that the primitive gnomonogram will take the following forms (shown in Figs. 27—34):

Cubic System (Fig. 27). A square with a side equal to *r*, the radius of projection;
Tetragonal System (Fig. 28). A square with a side greater or less than *r* accordingly as the angle 001 : 101 is greater or less than 45°;

Orthorhombic System (Fig. 29). A rectangle with the pole *c* (00), *i.e. indices* (001) at the centre (an example will be given later in connexion with the multiple tangent table, *q. v*);

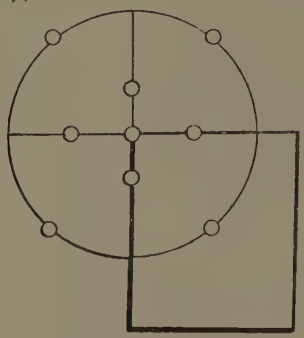


FIG. 29.

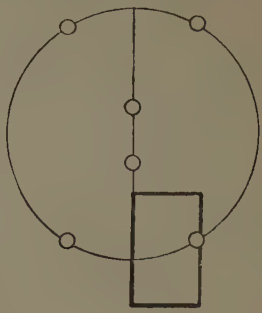


FIG. 30.

Monoclinic System. (1) A rectangle with the pole *c* (00) symmetrically displaced from the centre (Fig. 30), provided the crystal be projected with the axis *OZ* vertical; (2) A general parallelogram (Fig. 31) with the pole *b* (00) at the centre, provided the crystal be projected on the plane of symmetry. *In the latter case the two co-ordinates refer to the first and third indices, the second index being obtained by interpolating the number 1 (unity)*;

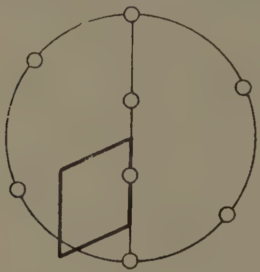


FIG. 31.

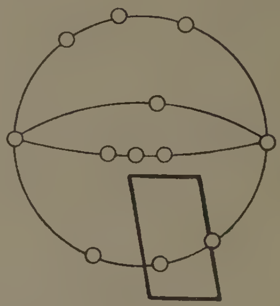


FIG. 32.

Anorthic System (Fig. 32). A general parallelogram with *c* (00) displaced from the centre;

Hexagonal System (Fig. 33). A 120° rhomb—by discarding the third index. The index discarded can always be inserted ultimately, from the rule that the sum of the first three indices (in Bravais' orientation) is equal to zero.

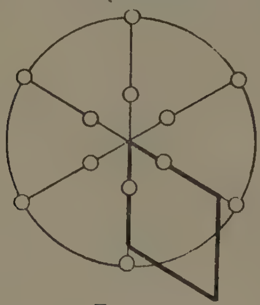


FIG. 33.

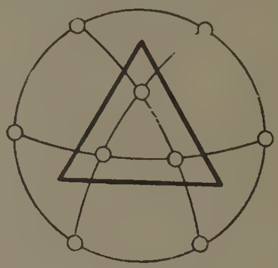


FIG. 34.

Rhombohedral System (Fig. 34).

In conclusion the opinion may be expressed that it is expedient to concentrate on the gnomonic projection in so far as it relates to the cubic, tetragonal, orthorhombic, monoclinic and anorthic systems because the system of symbolisation is uniform throughout, and because the last three mentioned systems are the systems of the future. The hexagonal and rhombohedral systems can well be reserved for a later stage of study, for although these two systems have played an important part in the past (especially in the province of mineralogy), their rôle in the future is not likely to be impressive. An analysis of various factors leads to the estimate that the number of hexagonal and rhombohedral structures to be incorporated into the roll of known crystals will not amount to two per cent. of the whole. With regard to the all-important monoclinic system there is no doubt that projection on the plane of symmetry, as usually taught to the elementary student, is a fine mental discipline (and the same naturally holds for the gnomonic projection), but the practice is to be deprecated in research work, for the unusual orientation of axes appreciably lowers the rate of work. [In so far as projection on the symmetry plane is advisable, the orientation of Fig. 31 appears to be more appropriate than that usually adopted, since the former is derivable from the 'anthropomorphic orientation' of Fig. 30 by a simple rotation about the normal to a (100), whilst the latter demands a *subsequent* rotation of 90° about the normal to b (010)—but, as already indicated, the present writer regards this question as of purely academic importance.] The orientation of Fig. 30 (here recommended) is of course that generally adopted outside England and France.

4. **Mallard's Gnomonic Theorem.**[6]—This theorem is more profound than that of Neumann, inasmuch as it penetrates beneath the surface into the structure. The basis of the theorem is the *polar lattice* of Bravais (not the 'ordinary' Bravais lattice). Mallard proved that in the special case of any face having the indices ($hk1$), the distance of the corresponding gnomonic pole (xy) from the point of projection O' (*i.e.* the point lying at a vertical distance r below the centre of the projection O) is a *comparative* measure of the reticular density (the massing of the structural particles) in the plane of that face. Consider Fig. 35, in which the gnomonic projection of a crystal of idocrase

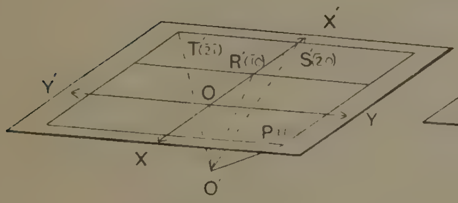


FIG. 35.

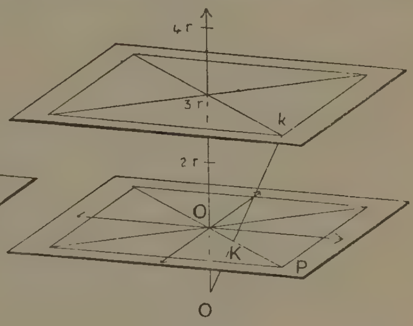


FIG. 36.

is once more shown in perspective. The centre of the crystal (*i.e.* the point of projection) is located at O', at a vertical distance r below the gnomonic plane. Then, according to Mallard's theorem the distance $O'O$, for example, is a measure of the reticular density in the structural plan c (001); $O'P$ is a measure of the reticular density in the plane $P(111)$; $O'R$ in the plane $R'(\bar{1}01)$; $O'S'$ in the plane $S'(\bar{2}01)$; $O'T'$ in

[6] This section as well as the next section on 'The Angle Point' may, perhaps, well be omitted on a first reading. In Fig. 36 the lowest point should be O'.

the plane $T'(\bar{2}\bar{1}1)$, and so on. In every case we have been dealing with faces which have *unity* as third index.

We may now consider a more general case. Suppose we have to deal with a face $K(113)$ [Dana's 'theta']. The gnomonic co-ordinates are $(\frac{1}{3}, \frac{1}{3})$; the pole therefore lies on the gnomonic zone-line OP at a distance from O which is equal to $\frac{1}{3}$ OP (*cf.* Fig. 36). Mallard proved that the reticular density is *not* measured by the length $O'K$, but by a length $O'k$ obtained by producing $O'K$ till it meets the upper plane of Fig. 36, which is placed vertically above the 'crystal centre' O' at a distance of $3r$. In general, if the point O' lies on the ground floor, the gnomonic projection on the first floor, and there be a series of upper floors at regular distances r, then the reticular density of a structural plane (hkl) is measured by the line drawn from O', through the point $(h/l, k/l)$ of the gnomonic projection (first floor) until it meets the lth floor (Floor No. 'l').

For the method of comparing reticular densities in the vertical faces, as also the special treatment necessary for hexagonal and rhombohedral crystals, Mallard's *Traité de Cristallographie* (1879) may be consulted.

Those who are interested in crystal structure will realise:

(1) that the lattice must be described in terms of the correct indices (in other words, the real structural lattice is assumed—not any lattice of the conventional crystallographic description);

(2) that if the lattice is not of the plain, pinacoidal type, certain reticular densities (discernible from a scrutiny of the indices—by noticing which and how many indices are odd numbers) must be modified (halved or doubled).

(3) that the X-ray crystal analyst's principal grating-distances, being the reciprocals of the reticular densities, can be conveniently compared with one another by the help of the gnomonic projection; or, alternatively, by numerical tables (based thereon) which have been compiled by V. I. Sokolov and D. N. Artemiev (*J. School of Mines, Petrograd* [Russian], 1909, **2**, 333; *Zeitsch. Kryst. Min.*, 1911, **48**, 377);

(4) that the Mallard theorem is one of the bases of Fedorov's method of identifying crystalline substances by virtue of their characteristic forms and structures.

Of the numerous cases in the history of science in which the publication of a really fundamental property has long been overlooked, the Mallard Theorem furnishes a conspicuous example. The case is, perhaps, all the more remarkable inasmuch as the theorem was published in a general text-book and not in the pages of an inaccessible journal. Amongst those who had a use for it (if they had known of its existence) was Fedorov, who might thereby have been spared much labour in connexion with the development of 'crystallo chemical analysis'—for it appears to be certain that he would have abandoned the principle of simple indices for the Bravais principle (*cf.* p. 107) earlier than he did, if the theorem had been close to hand. As it was he finally deduced the theorem independently (*J. School of Mines, Petrograd* [Russian], 1908, **1**, 279) and

erected on it his final classification of the crystal kingdom. The present writer would add that he only became aware of Mallard's theorem two years ago, in studying the *Traité*, and had previously regarded it as the 'Fedorov Theorem.'

5. **The Angle Point.**—In the stereographic projection the angular distance between two face-poles can be measured by projecting lines from the zone-pole to the primitive. A somewhat similar, and equally direct, procedure can be adopted with the gnomonic projection, the zone-pole being replaced by a special 'angle-point,' W (in German 'Winkelpunkt').

In Fig. 37, O' represents the crystal-centre; E, the gnomonic plane; $O'O = r$, the normal to the plane; $O'A$ and $O'B$, the normals to two faces, so that the angle $AO'B$ necessarily represents the true interfacial angle AB. The following argument is merely designed to elucidate the nature of the angle-point, W. The actual construction will be given later.

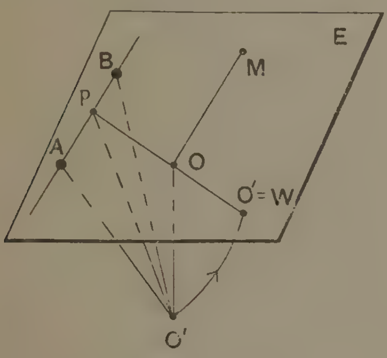


Fig. 37.

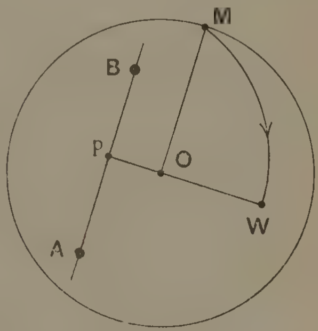


Fig. 38.

Draw Op, normal to the zone-line AB. Join $O'p$. Note that $O'Op$ is a right-handed triangle, of which $O'p$ is the hypotenuse. Moreover, the triangle $O'Op$ is perpendicular to the triangle $O'AB$; whence it follows that if the latter be turned about AB as hinge until it coincides with the plane E, then the line pO' will coincide with pO produced, and the angle AWB (where W has been substituted for O' in its new position) will represent the true angle AB required. The point W therefore lies on pO produced. Now suppose the triangle be hinged about pO; the point O' will swing up into M, where OM is parallel to ApB, and is equal to r. We have now all the data for a simple construction.

Problem I. **Given a Gnomonic Zone-line to Measure an Interfacial Angle.**—In Fig. 38 representing the *actual* projection A and B are the two poles. From O draw the normal to this zone-line and also a parallel line OM. With centre p and radius pM strike the arc MW. Then W is the angle-point. Lines can now be drawn from A and B to

W, and the angle *AWB* measured by means of the semi-circular part '*C*' of the crystallographic protractor.

It must now be pointed out that the measurement of angles in the gnomonic projection is not to be generally recommended to the single-circle worker, for he is only adding one error on to another (the projection having been obtained *indirectly* from the measurements). The angle-point has, however, a supreme importance in crystal drawing, and it is therefore worth while mentioning that *W* can be obtained (without striking the arc *MW*) by means of the main edge '*A*' of the protractor. Measure *Op* gnomonically $= x°$; then *OW* is stereographically $(90° - x°)$.

Problem 2. **To Measure the Angle between Two Gnomonic Zones.**—The two zone-lines are represented by *A* and *B* in Fig. 39. They intersect in the point *P*, but it is important to note that the angle required is not simply the plane angle *APB*, for unlike the stereographic *the gnomonic projection does not preserve angular truth.* This does not mean that the angle *AB* cannot be determined accurately. The method is perfectly analogous to that of Problem 10 of the previous chapter: we replace each zone-line by its pole and measure the angle between these two poles. As in the stereographic, so in the gnomonic projection a pole is on the other side of the centre at a distance of 90°

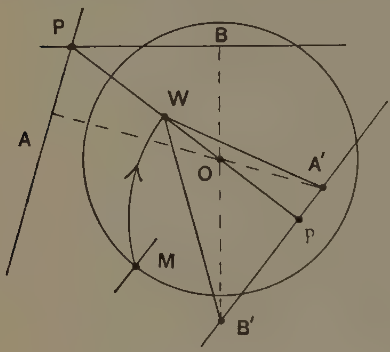


FIG. 39.

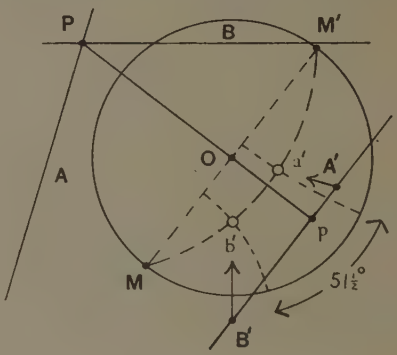


FIG. 40.

(measured *gnomonically* of course in the present case), and it follows that if a gnomonic zone-line lies outside the primitive circle, its pole will be inside; and vice-versa. To locate the pole of *A* we place the main edge of the protractor on the interrupted central line normal to *A*, observe the gnomonic reading *OA* and mark off *A′* its gnomonic complement. Similarly *B* gives its pole *B′*. The angle *A′B′* is the angle required. In order to determine its value we have to locate its angle point *W* by the method of the preceding problem, and measure the plane angle *A′WB′*. It is interesting to note that *W* and *p* must theoretically lie on the line *OP*, and that *p* is the pole of *P* (*i.e.* is

distant from it 90° gnomonically). These relationships can be used as a check on the correct dispositions of A' and B'.

NOTE.—It is frequently advantageous to carry out a determination partly in the gnomonic and partly in the stereographic projections. In the present case we might usefully proceed to the stereographic as soon as the gnomonic poles A' and B' have been determined (*cf.* Fig. 40). Thus, we successively place the main scale along OA' and OB' and plot in the corresponding stereographic pole a' and b' at half the respective angular distances. We have now to determine the angle $a'b'$ measured along a great circle (shown as an interrupted arc in Fig. 40). These is no need to draw this circle, although it is so easily effected by taking P as centre and PM as radius (another example of the inseparable nature of the gnomonic and stereographic projections). We simply mark in diametral points M,M' parallel to $B'A'$ and align the part C of the protractor along this diameter. In the original drawing it was thus seen that the meridian $a'b'$ is a perfect 37° meridian (a sign of accuracy in the procession from the gnomonic poles $A'B'$ to the stereographic poles $a'b'$); also that the interfacial angle $a'b' = 51\frac{1}{2}°$. This is the true angle between the zone-lines AB, however much the apparent angle APB deviates therefrom (in this case being $73\frac{1}{2}°$).

Problem 3. **To Determine a Zone-Symbol from the Gnomonic Projection.**—Theoretically, zone-symbols are best derived graphically from another projection, generally termed the 'linear'; practically, the following method due to Mallard is just as effective (*Traité de Cristallographie*, pp. 65-66). A Figure need not be given here. Suppose we have a zone [$\bar{2}11$:012]. Required its zone-symbol by some way different from that of cross-multiplication. We note the two points 2 and $\frac{1}{2}$ in which the gnomonic zone-line crosses the co-ordinate axes. We have now to take their reciprocals, namely $\frac{1}{2}$ and 2, and add *minus* 1 as third index, thus $\frac{1}{2}2\bar{1}$, and clearing of fractions obtain [$14\bar{2}$]. It is not here suggested that the method is superior to the method of cross-multiplication; the latter method is the better, if one is operating exclusively with the stereographic projection (or with no projection). [The real interest lies in the method by which Mallard deduced it—from his gnomonic theorem.]

6. **Preparation of the Gnomonic Projection from Two- and Three-Circle Measurements.**—In the case of single-circle measurements the gnomonic projection should always be prepared from the stereographic—in order to avoid constructions involving more general conic curves than the circle. With regard to two-circle measurements we must again distinguish between the zone-adjustment and the face-adjustment. With the former (*cf.* Fig. 15, p. 12) it is clear that if we plot the gnomonic ρ-reading instead of the stereographic, the gnomonic projection is obtained. This means that the projection can be prepared

more accurately than in the case of single-circle goniometry. With the face-adjustment, on the other hand, the projection has to be prepared *either* through the stereographic *or* by a more accurate method given by G. F. Herbert Smith (*Min. Mag.*, 1903, **13**, 309; *Zeitsch. Kryst. Min.*, 1904, **39**, 142) who has computed a table from which the rectangular co-ordinates of a gnomonic pole can be immediately read from the ϕ and ρ-readings (in the sense of Fig. 16, p. 12)—the use of squared paper (though not necessary) is naturally of assistance. As mentioned previously, three-circle goniometry is a generalised face-adjustment type of two-circle work, and the method just outlined was, as a matter of fact, designed for this generalised form of measurement.

7. **Some Disadvantages of the Gnomonic Projection.**—The gnomonic projection suffers from certain minor and major disadvantages. In the first place the poles of the vertical faces lie at infinity. Secondly, the poles of faces, which approach the vertical, lie at an inconveniently great distance from the centre. A reduction of scale only brings a partial relief, for the projection as a whole then becomes too minute. The next disadvantage is much more serious. In a stereographic projection a small circle on the sphere, expressing the locus of points x° distant from a given pole, is projected as a small circle. In the gnomonic projection this is only true if the pole coincides with the centre of the projection; in other cases the projected locus is an ellipse or other conic section, not so easily and accurately drawn as a circle. Accordingly any such construction, as involves the intersection of two small circles in the stereographic, is scarcely practicable in the gnomonic without special apparatus. The best auxiliary would seem to be the gnomonic net devised by H. Hilton (*Min. Mag.* 1904, **14**, 18).

8. **Advantages and Disadvantages of the Stereographic Projection.**—The first advantageous property of this projection is that one half of the crystal (say the upper half) can be projected within the limits of the primitive circle, and that the crystal as a whole can be projected on a given finite piece of paper to exactly the same extent as the upper (or lower) half can gnomonically. The second is that, although zones are not projected as straight lines, both zones and small circles are nevertheless projected as circles—figures that can be drawn with great accuracy provided the radius be not too great. This means that any such protractor as part *C* which consists of a series of engraved meridians and small circles (or any corresponding pattern of curves on paper—' stereographic net ') can be prepared relatively inexpensively. The two properties combined make the projection of unique value to crystallography for general illustrative purposes, and also confer on the projection a measure of accuracy (whenever numerical work is concerned) which is only exceeded by the gnomonic. The main dis-

advantage is that it does not allow of a direct determination of indices, or of a solution of manifold structural problems.

9. **The Advantages of the Combined Projection.**—The simple angular relation between the two projections (the gnomonic and stereographic poles of terminal faces being necessarily on the same radius, the former at *twice* the angular distance from the centre) makes it quite easy to combine the two projections. This does not mean that both have to be prepared in every case; it merely implies that if we have one, or the other, or parts of one amplified by the complementary parts of the other, then we can proceed from one to the other accordingly as the nature of the problem demands. Thus, if working stereographically, we suddenly have to deal with a meridian circle of long radius and do not possess the necessary arc-ruler or beam compasses we can proceed to the gnomonic zone-line (and angle-point) for the solution and finally revert to the stereographic. Conversely, if in working gnomonically, we have to insert a pole from a pair of angular readings, the proper course is to proceed to the stereographic (in order to make use of circles in lieu of more elaborate conic sections) and finally revert to the gnomonic. These are illustrations of the truth that (in a perverted sense excepted) disadvantages no longer exist in any combined projection. There are, so to speak, no poles at infinity—since they lie in the 'stereographic primitive' or 'gnomonic unit-circle.' [7]

The initial projection in crystal measurement must naturally be the stereographic, because the first real problem in the investigation of a new substance is the determination of the system. As is well known this involves the comparison of angles, and angles can be measured graphically more directly in the stereographic projection than in the gnomonic (in this connexion it only seems necessary to recall the use of the engraved circles of the part *C* of the crystallographic protractor). It is only in subsequent operations that the gnomonic projection insistently challenges the notice of the researcher. Accordingly, the two projections will begin to play their equal rôle in the following pages.

EXERCISES.

Exercise 4.—From the stereographic projection of Exercise I proceed to the gnomonic by laying off twice the central angular distance of every terminal pole. Draw the primitive gnomonogram, (001)—(101)—(111)—(011). Reduplicate and subdivide it as necessary in order to see how closely each gnomonic pole corresponds to its theoretical position in the simple cases, (121), (131) and (221). Determine numerically the

[7] Throughout this book a practice will be made of inserting the stereographic poles of the vertical faces on this unit-circle: in the first place, as a useful reminder of the virtual presence of the stereographic projection in any gnomonic diagram; and, secondly, because the enclosure of the latter in an essentially artificial frame, containing the directions of poles lying at infinity, is thereby obviated.

plane co-ordinates for the gnomonic poles x and y, and by adding unity as third index and clearing of fractions determine their indices.

Exercise 5.—Prepare the primitive gnomonogram, (001)—(101)—(111)—(011) for the tetragonal anatase, given the value 001 : 101 = 60° 38′. Divide the diagonal (001)—(111) of the square into thirds, and with the main scale of the protractor measure the angles (001) : (113), (001) : (223) and (001) : (111). Locate the gnomonic poles of (335) and (256) by means of their plane co-ordinates h/l, k/l, and measure the angle each makes with (001), or otherwise stated, their rho-values.

Exercise 6.—Proceed to the gnomonic from the stereographic of Exercise 2. Prepare the elementary gnomonogram (001)—(101)—(111)—(011), contrast it with the idocrase gnomonogram, and determine the indices of the faces, i, u, y, v, and w.

Exercise 7.—Prepare a general stereographic projection (with symmetry plane vertical) of the forms (001), (110), (011) and (101) of the monoclinic iron vitriol (melanterite) given the angles $\beta = 104°16'$, $(001) : (101) = 43°44'$, $(010) : (011) = 33°47'$ and $(010) : (110) = 41°6'$. Proceed to the gnomonic poles of (001), (101) and (011), and construct therefrom the primitive gnomonogram, (001)—(101)—(111)—(011). Measure the rho-value of the gnomonic pole (111). Reduplicate the gnomonogram, to mark in the gnomonic poles of ($\bar{1}$01), (013) and (112) and measure their rho-values. Mark in the gnomonic pole of x(121). Measure the angle (101) : (121) in both the following ways : (a) gnomonically, by first finding the angle point of the zone; (b) stereographically, by proceeding from the gnomonic pole of x to its stereographic pole, and measuring the zonal arc by means of the part C of the protractor. Repeat this exercise, if necessary, on a later occasion.

**Exercise* 8.—Prepare (without necessarily drawing any zones save the primitive) a stereographic projection of anorthite exhibiting the following forms, (010), (110), (1$\bar{1}$0)), (111), (1$\bar{1}$1), ($\bar{1}\bar{1}$1) and ($\bar{1}$11) from the following two-circle data (in which the first of each pair of angles quoted represents the phi-value, and the second the rho-value (in the sense of Fig. 15 of the previous chapter) :—

(010), 0°0′, 90°0′; (110), 58°4′, 90°0′; (1$\bar{1}$0), 117°33′ 90°0′; (111), 64°50′, 58°2′; (1$\bar{1}$1), 106°13′, 56°29′; ($\bar{1}\bar{1}$1), 222°35′, 35°12′; ($\bar{1}$11), 320°44′ 37°1′. From the stereographic poles of the four terminal faces proceed carefully to the gnomonic poles. Draw the quadrupled primitive gnomonogram, and subdivide it into its four component parts. Prolong the gnomonic zone-line, (0$\bar{1}$1)—(001)—(011) and plot on it the gnomonic pole of the commonly observed face e(021). Measure the angle (001) : (021) gnomonically by the angle-point construction. From the gnomonic poles (001) and (021) proceed to the corresponding stereographic poles. Draw the zonal arc (010) : (001) and measure the angle (001) : (021) stereographically.

CHAPTER IV.

THE GRAPHICAL DETERMINATION OF INDICES.

'Die diesen Flächen entsprechenden Flächenorte wird man leicht aus der gewählten Bezeichnung derselben ersehen können; es ist nähmlich der Ort von der Fläche (a/m : a/n : c/p) bezeichnet durch n/p (m/p),—so dass die beiden Ordinaten $(n/p)\beta$ und $(m/p)\alpha$ des Punktes ihn bezeichnen. So kann mann immer unmittelbar den Flächen-Ausdruck weider aus dem Schema ablesen.' F. E. NEUMANN (1823).

[Or in modern phraseology: The gnomonic poles corresponding to these faces will be easily picked out from the diagram by means of their characteristic symbols, for the pole of a face (hkl) is designated h/l, k/l, its plane-co-ordinates being h/l and k/l *times* the unit lengths. And it is therefore possible to read the indices directly from the diagram.]

After a stereographic projection has been prepared and the system determined, the next step in crystallographic routine is the determination of indices. In many cases this scarcely constitutes a problem, for the indices are obvious. Thus, all forms which can be symbolised by permuting the indices (100), (110) and (111) require no elucidation. In other cases, the determinations, as generally practised at the present day, follow classical lines, the simpler indices being determined by 'cross-multiplication,' and the more complicated by the use of Miller's well known formulae, connecting angles and indices of four tautozonal faces (or coplanar zone-axes). Graphical methods, on the other hand, would seem to be generally neglected, although they can be used with great advantage whenever several series of cross-multiplications have to be effected, or whenever even a single evaluation of the Millerian formula is the only other alternative.[8]

It is convenient to distinguish two graphical methods of determining indices: the gnomonic and the stereographic, although the latter would seem to be a variant of the former. The former is the more general, and should be held in reserve for the more complex cases. The latter is especially useful where the problem is narrowed down to the treatment of a single zone (particularly the vertical zone). As space does not permit a consideration of all the systems of crystallisation, we will take examples from the anorthic system as being the most general and difficult.

[8] An exception to this statement must be made in the special case of the rectangular zone, for the indices can generally be read from the multiple tangent table (see p. 67).

I. THE GNOMONIC METHOD.

The example chosen is the penta-hydrated copper-sulphate—the mineral chalcanthite. The description will be based on Groth's orientation (*Chemische Krystallographie*, **2**, 419). It should be noted that the orientation is different from that adopted by Dana, and that Goldschmidt uses a different set of indices, while Artemiev (in a Russian paper) uses still another form, based on Fedorov's ideas on the 'correct setting.' In order to render a reader independent of all these original sources, the relevant angles will be introduced into the following account. Capital Roman letters have been substituted for Groth's small Greek type.

It will now be well to enumerate the indices in order to make the description clear. The forms developed are :—$b(010)$, $m(110)$, $a(100)$, $M(1\bar{1}0)$, $P(1\bar{3}0)$, then, $t(021)$, $q(011)$, $K(0\bar{1}1)$, $T(0\bar{2}1)$; and finally, $Z(\bar{1}31)$, $X(\bar{1}21$, $O'(\bar{1}11)$ and $S(\bar{1}\bar{2}1)$.

1. **Preparation of the Stereogram.**—The vertical faces are marked in the primitive (Fig. 41) from the angles $bm = 52°59'$, $ba = 79°6'$, $bM = 110°8'$, $bP = 146°10'$. The face-pole of a terminal face q is then located (by the help of the edges 'B' and 'C' of the protractor) by the intersection of two small circles; $bq = 64°58'$, $aq = 69°59'$. The inclination of the zone bqb' (not yet drawn) is now measured by the help of the meridians engraved on 'C'; found $16\frac{1}{2}°$. The protractor is now placed with its main edge 'A' perpendicular to bOb' and the centre of the zone circle (off the confines of Fig. 41), distant $73\frac{1}{2}°$ gnomonically

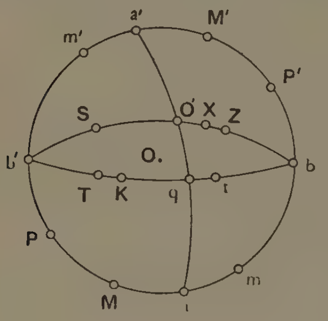


FIG. 41.

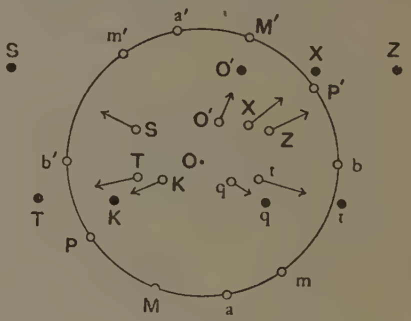


FIG. 42.

(= 147° stereographically), and also its pole, distant $73\frac{1}{2}°$ stereographically, are simultaneously noted. The zone circle bqb' can now be drawn, and the various face poles, t, K and T inserted, by laying off the angles $bt = 44°41'$, $bK = 121°57'$, and $bT = 139°27$ on the primitive, and projecting them to the zone-pole. Attention is now centred on the zone aqa' (not yet drawn). Its inclination is measured by means of 'C' (found $22\frac{1}{3}°$), its centre and pole located by the main edge 'A,' and the face pole O' marked in from the measured angle $a'O' = 59°25'$.

The zone bO' (not yet drawn) is now examined. Its inclination is found by means of 'C' to be $34\frac{1}{3}°$, its 'centre' and pole located as before by the main edge 'A,' and the various face-poles marked in from the data $bZ = 40°56'$, $bX = 54°44'$ and $bS = 139°8'$.

2. **Preliminary Allocation of Indices.**—For purely descriptive purposes an original author has of course a free choice in the anorthic system; he could, for example, allocate the indices (001) to the face q. The indices here adopted are $q(011)$, $K(0\bar{1}1)$, $O'(\bar{1}11)$, $a(100)$ and $b(010)$—the basal plane $c(001)$, being rarely developed. This completes the 'fundamental' allocation of indices. As there are numerous faces to be determined, the problem is evidently a suitable subject for the gnomonic probe.

3. **Transformation of the Stereographic Projection into the Gnomonic.**—This is effected for each terminal face by means of the main scale 'A' of the protractor. The edge is aligned with each radius carrying a stereographic pole, the angular distance carefully noted, and a point having twice this angular distance marked off (in other words, the gnomonic graduation is substituted for each stereographic graduation). Nothing can be done with the stereographic poles lying on the primitive (since the corresponding gnomonic poles lie at infinity). The combined gnomono-stereogram is shown in Fig. 42, in which the zonal arcs have been omitted. It should, perhaps, be added that the actual work from start to finish (Figs. 41—43) should be done in one diagram. There is no confusion after a little experience has been gained.

4. **The Terminal Faces.**—The next task is to establish the primitive gnomonogram and reduplicate it as far as is necessary. We meet with a small difficulty at the outset; the basal plane $c(001)$—the origin of co-ordinates—is not actually developed, but, as the simple linear properties of the gnomonic projection demand an equality of spacing along the zone Kq, we have only to join Kq (see Fig. 43) and bisect this distance in order to obtain the origin of co-ordinates $c(00)$. The line Kq is of course the axis $Y'OY$; the axis XOX' can be obtained by drawing a line through $c(00)$ parallel to the zone-line qO' (compare the similar case of Fig. 26). The primitive gnomonogram (in this case in the back right-hand quadrant) follows immediately from the three corners c, q, O'. It is now carefully reduplicated; the coordinates of the various poles are inscribed; and the indices read off by adding unity as third index in every case. It will be seen that there is no ambiguity although the gnomonic poles were not derived directly from the measurements (as can be done with two-circle goniometry) but indirectly from the stereographic projection.

5. **The Vertical Faces.**—The indices of these faces, all having the general form $(hk0)$, have perforce to be determined from the stereo-

graphic poles. The determination is easily effected by the following method. Each stereographic pole (*hko*) is related to its 'directrix,' the gnomonic pole of the corresponding (*hk*1) in the following respect: if the radius carrying the pole (*hko*) be translated till it passes through the centre of co-ordinates *c*(00), it will simultaneously pass through the point (*hk*), *i.e.* through the gnomonic pole of (*hk*1). If, then, the indices of *m* have to be determined, the edge of a set-square is aligned with *Om* and then *translated* (by the help of a second set-square) *until the line passes through c*(00); its course now lies very close to the

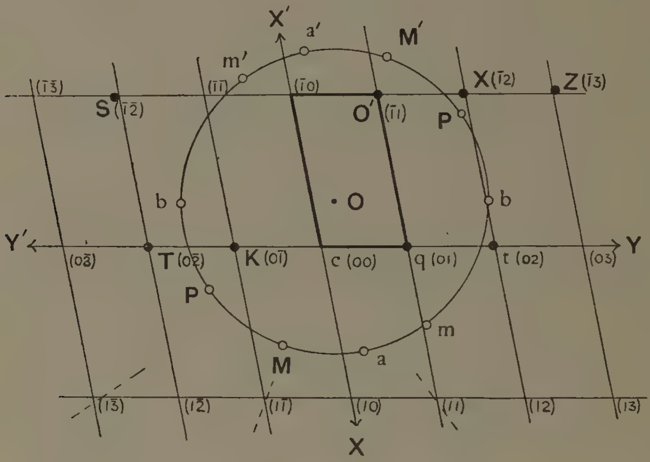


FIG. 43.

gnomonic pole (11), as is shown in Fig. 43 by an interrupted line. Adding zero as third index we obtain (110) as the indices of *m*. The corresponding, determinative interrupted lines for the faces *M* and *P* are also shown: In every case we have to noted where the *translated radius* intercepts the gnomonic zone line [101 : 111]. As the points are very close to (11̄) and (13̄) respectively the indices are *M*(11̄0) and *P*(13̄0).[9]

6. **The Most Accurate Method of Establishing the Primitive Gnomonogram.**—It is of great practical importance to note that the simple relation between a stereographic pole (*hko*) and the gnomonic pole of its directrix (*hk*1), makes it possible to use a vertical face in establishing the primitive gnomonogram. In fact, the vertical faces should be preferred, because they are more accurately placed in the projection. Suppose the zone lines *Kq* and *O′q* were already drawn;

[9] It is important to note that, in systems like the orthorhombic, in which the gnomonic pole of *c* (001) coincides with the centre of the primitive, *there is no need to translate the radius*, for in such a case a radius drawn through a stereographic pole (*hko*) already passes through the directrix (*hk*). The following section, on 'The most accurate method of establishing the primitive gnomonogram,' may well be omitted on a first reading.

we can use $m(110)$ to determine the unit length along $O'q$—simply by translating the radius Om till it passes through the centre of co-ordinates $c(00)$. The point in which this translated radius cuts the zone $O'q$ *is* the true unit (positive) distance; but the most accurate way of establishing the elementary gnomonogram, in the case of copper sulphate, is to base it on the poles $a(100)$, $m(110)$, $b(010)$, $K(0\bar{1}1)$ and $q(011)$—these faces, of course, having been previously taken to have these indices. We first derive the gnomonic poles of K and q from the stereographic. Now the axis YOY', that is the zone line Kq, *must* be parallel to the diameter bOb'. We therefore draw a line parallel to bOb' through K and q, halving any discrepancy there may be. As before, the line Kq is then bisected at c. Then, again, the axis XOX' must be parallel to the diameter aOa'. We therefore draw through K, c and q a series of lines parallel to the diameter aOa'. The unit length along the q-parallel is then obtained as described above from $m(110)$, and may be checked and adjusted by similar operations relating to the face $M(1\bar{1}0)$ and the *K-parallel.*

7. **The Converse Problem.**—This heading is merely added to remind a reader that if the indices of a face are given its gnomonic pole (in the case of a vertical face, its directrix) can be inserted in the projection by the help of its plane co-ordinates, and immediately transformed into a stereographic pole. In the case of a terminal face the first and second indices are of course first divided by the third, the values so obtained being the plane co-ordinates required.

8. **The Reliability of the Gnomonic Method.**—In the case under consideration there is no ambiguity whatsoever in the value of the plane co-ordinates: Z $(\bar{1}3)$ and S $(\bar{1}2)$ are a little out, but there is no reasonable doubt in view of the inevitable small errors that the indices are $(\bar{1}31)$ and $(\bar{1}21)$ respectively. The question now arises whether there is ever any danger of deriving the wrong indices. (This question is bound up with the degree of perfection of the tools and the care and skill of the workman, but these will not be discussed.) A glance at the gnomonic diagram will be sufficient to indicate the inevitable source of error if the method be pushed too far. If the 'real indices' be so complicated that the derivation of the plane co-ordinates demands, either a drastic sub-division of the primitive gnomonogram, *or* its excessive reduplication (or, worse still, *both*), then ambiguity will surely arise—especially in the rare case in which the central angular distance of a pole exceeds 70°, for an error of ½° now makes a considerable difference in the *gnomonic* scale (an illustration of this will be given presently). It will be realised that no hard and fast rule can be formulated, because the size of the primitive gnomonogram comes into play. Experience shows that with the arrangement under discussion (indirect

preparation of the gnomonic from the stereographic projection and the use of the present protractor of 5 cms. radius), ambiguity is to be feared when a number exceeds 6. This means that the indices of faces involving the number 7, or a higher value, cannot have their indices determined with certainty in this way, and some form of calculation must be invoked. This is really a most satisfactory conclusion, for crystals exhibiting facets with one or more index numbers greater than 6, form less than 3 per cent of the crystal kingdom. They almost wholly belong to the province of mineralogy (see special note at the end of this chapter), *i.e.* to a category of substances which have grown under conditions in which periods of corrosion may have alternated with periods of growth, the result being the occasional development of corrosion surfaces or planes with high indices. If minerals be left out of account, the number of cases which can be properly solved by the gnomonic projection rises to $99\frac{1}{2}$ per cent. There are, indeed, but few crystallographers who have ever encountered a single case in the province of laboratory products, in which the gnomonic projection is not perfectly competent to determine all the indices. The full significance of this will only appear in Chapter VIII.

The following illustration of the limits within which the gnomonic method can be trusted, although of considerable importance, may perhaps be omitted on a first reading on account of its complexity. The example is taken from a paper by V. Rosicky (*Zeitsch. Kryst. Min.*, 1909, **46**, 357; Groth's *Chem. Kryst.*, **3**, 54) containing a description of an isomorphous series of double sulphates of ethylene diamine and manganese, iron and cadmium, with the general formula

$$C_2H_4(NH_3)_2SO_4,RSO_4,4H_2O,$$

which crystallise in the anorthic system. The form development is unusually complicated, consisting of $c(001)$, $b(010)$, $m(110)$, $M(1\bar{1}0)$, $p(133)$, $P(1\bar{3}3)$, $d(\text{-}8, 0, 3)$, $f(4, 12, 3)$, $F(4, \bar{1}\bar{2}, 3)$, $r(\bar{2}63)$, $R(26\bar{3})$, and $E(\text{-}8, \text{-}24, 3)$. A combined gnomono-stereographic projection is shown in Fig. 44, the stereographic having been first prepared from Rosicky's single-circle readings by the help of the protractor, and the gnomonic by marking off twice the central angular distance of each pole. The gnomonic poles of d and E lie far outside the limits of the drawing. The primitive gnomonogram was then constructed as follows. The gnomonic poles of c and d (the latter off the present diagram) were joined, and a distance equal to three-eights marked off to represent one side of the gnomonogram. Through the terminal points of this side were drawn two lines parallel to the diameter $b'Ob$, and the unit length of the far side was determined by translating the radius OM' till it passed through the gnomonic pole of c. The parallelogram was then completed and reduplicated. In the original drawing the reduplication

was of course carried out far beyond the confines of Fig. 44, in order to determine the plane co-ordinates of the gnomonic pole of *E*, the co-ordinates of which were thus determined to be -3, -9 corresponding to indices (-3,-9,1). *This result was erroneous* for the plane co-ordinates should have been -8/3,-8, since the true indices of *E* are (-8,-24,3).

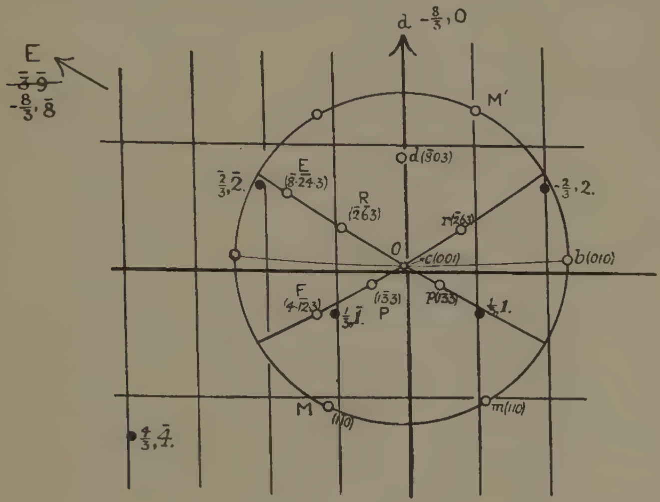


FIG. 44.

A subsequent search for the source of error revealed the fact that there was an error of 2/3° in the position of the stereographic pole of *E*, and that an error of 1/3° had been made (in the same direction) in the derivation of the gnomonic pole from the stereographic. This total error of 1° is enough at this great angular distance from the centre (about $77\frac{1}{2}°$) to make a linear difference which almost amounts to the shorter side of the primitive gnomonogram. With regard to the remaining poles, the agreement is more than satisfactory. There is, *e.g.*, no difficulty in determining the plane co-ordinates of *F* to be 4/3, -4, and the indices to be (4, $\bar{1}\bar{2}$, 3).

NOTE (added in proof-correction).—Neither the stereographic nor the gnomonic pole of the face *f*(4, 12, 3) is included in Fig. 44, because it was based on the description of the cadmium salt. I now recognise that it would have been better to have based it on the data given for the iron salt, but I have satisfied myself by a process of calculation followed by an insertion into the original pencilled diagram of the imaginary pole *f* that its indices would have been unambiguously determined by the gnomonic method. In actual practice the determination of *f* gnomonically would have been superfluous, for considerations of pseudo-symmetry may be invoked to derive the indices of *f* from *F* (the same being true of *m* and *M*; *p* and *P*; and *r* and *R*).

The above results, including a possible erroneous determination of *E*, were expected from previous experience, but the case is none the less instructive as illustrating the necessity for caution whenever a pole lying at a great central angular distance is concerned. The example is also interesting in another direction. Substances exhibiting faces of

such complexity are of exceedingly rare occurrence in the field of laboratory products; in fact, the form *E* (-8, -24, 3) is unique. In the present case the complexity is apparent rather than real. It was pointed out by Rosicky that the projection is that of a pseudo rhombohedral crystal. If we follow out this suggestion to its logical conclusion and effect a general transformation of the indices by means of the equations:

$p = -3h + 3k + 4l$; $q = -3h - 3k + 4l$; $r = 6h + 4l$, where (pqr) are the new indices of a face and (hkl) the old, the list of forms becomes: $c(111)$, $b(1\bar{1}0$, $m(0\bar{1}1)$, $M(\bar{1}01)$, $p(101)$, $P(011)$, $d(11\bar{1})$, $f(1\bar{1}1)$, $F(\bar{1}11)$, $r(100)$, $R(010)$ and $E(\bar{1}3\bar{1})$, and nothing could be more striking than the simplicity of these indices. They are, indeed, so simple that they can be read by the inspection of a *stereographic* projection.

II. THE STEREOGRAPHIC METHOD.

So far as a vertical zone is concerned this method appears to have been first introduced into crystallography by Fedorov in one of his early papers. It is also mentioned with a little amplification in a later paper. The method was independently applied by Moses and Rogers in 1902.[10] No proof is given by these authors, but one may be readily obtained from Miller's well-known formula, for the method is nothing more nor less than a graphical evaluation of the cotangent formula, discussed in Chapter VI. The method is also applicable to other zones, as will be made clear later by a further consideration of the gnomonic projection.

1. **The Vertical Zone.** Given the vertical zone of an anorthic crystal and the positions of the faces $a(100)$, $m(110)$, and $b(010)$. It is required to establish the indices of any other faces in the zone. Draw a line in any convenient situation parallel to the diameter bOb' (see Fig. 45) and also radii through $a(100)$ and $m(110)$. With the dividers repeat the intercept 0–1 (in Fig. 45 these intercepts are numbered to facilitate the explanation). Then a radius through 2 locates the stereographic pole of (120); through 3, a pole (130); through -1, a pole $(1\bar{1}0)$; through -2, a pole $(1\bar{2}0)$; through $\frac{1}{2}$, a pole (210); through $-\frac{1}{3}$, a pole $(3\bar{1}0)$; and so on. In general, a radius through the point k/h locates the pole $(hk0)$—*notice the inversion, usual to the*

[10] E. S. Fedorov, *Mining Journal* [Russian], 1887 [the writer has only a separate copy, in which the method is described on p. 34]; *Journal of the Petrograd School of Mines*, [Russian], 1912, **3**, 141. A. J. Moses and A. F. Rogers, *School of Mines Quarterly*, New York, 1902, **24**, 1 [reference not certifiable]; *Zeitsch. Kryst. Min.*, 1904, **38**, 209. A certain degree of caution is advisable in making any definite statement about the origin of a graphical method, for cases are particularly common in this province of crystallography, in which the same method has been independently deduced, or introduced as the case may be, by different workers independently. It would be hazardous to guess how long the principles of the method have been known to mathematicians.

relation of intercepts and indices. The pole $M(1\bar{1}0)$ would naturally have served equally well in establishing the unit intercept. Or, again, the poles m and M would have given the double value; the poles $m(110)$ and $n(120)$ the unit value; and so on.

The line could have been drawn parallel to the diameter aOa' instead of bOb'; in which case there would have been no inversion of numerical co-efficients (indices), but one of algebraic signs.

The converse problem admits of the same ready solution. Given the indices of any fourth face, its stereographic pole can be placed forthwith into the projection with a high degree of accuracy.

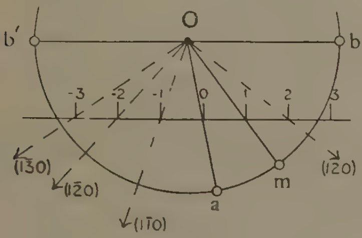


FIG. 45.

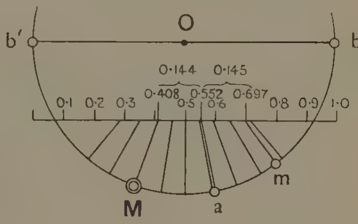


FIG. 46.

The above operations can be effected without any constructions, by the help of the segment '**E**' *of the crystallographic protractor.* The protractor is placed with the main scale accurately aligned with the diameter $b'Ob$, whereby the engraved segment coincides with the primitive (a much simplified representation is given in Fig. 46). The 'convenient line' mentioned above is both engraved and graduated in millimetres on the protractor. Moreover, radii are engraved every 5° within the limits 30°—150°; intermediate radii can be estimated. The figure refers to copper sulphate. Suppose the poles $m(110)$ and $a(100)$ have been marked in from the values $bm = 53°$, $ba = 79°$. To find the indices of a fourth face M (inserted in the projection from a measured angle $bM = 110°$ approx.). The estimated radii through M, a and m meet the scale in 0·408, 0·552 and 0·697 respectively (the third significant figure being doubtful). The differences 0·144, and 0·145 show an equality of intercepts. The indices are, then, $M(1\bar{1}0)$. The third significant figure is somewhat refined, two figures being sufficient. Thus, working quite rapidly, we have 0·41, 0·55, 0·70; the differences 0·14, 0·15 are sufficiently close. Crystals do not usually exhibit a face $X(15\cdot\overline{14}\cdot0)$ instead of $M(1\bar{1}0)$. A crystal obeys a law of *simple* indices.

The segment '*E*' can naturally be used with primitives of other radii than 5 cms. It can also be used without any projection whatsoever—granted a certain degree of familiarity with the method. Sup-

pose, for example, we are given the following data, $bx = 95°$, $ba = 79°$, $bm = 53°$. Required the indices of x. An inspection of the protractor shows that the radii meet the scale in 0·48, 0·55 and 0·70 respectively. The differences are 0·07 and 0·15, and the first is seen to be approximately half the other. The face in question is $(2\bar{1}0)$. Or, again, with the converse problem. Given $ba = 79°$, $bm = 53°$, and a face $l(120)$. Required the angle bl. The radii through a and m are seen to meet the scale at 0·55 and 0·70; difference = 0·15. As the face is $l(120)$ we have to advance to the right by one more step: 0·70 + 0·15 = 0·85, and an inspection of the scale shows the angle bl is approximately 37°. The measured value quoted by Groth is 37°18′; the computed value is 37°14′. It is possibly a favourable case; no other example has been tried.

2. **The Inclined Zone.**—Any inclined zone can be readily rotated over into the primitive by means of the part '*C*' of the protractor (or less rapidly by projection from the zone-pole), and then 'developed' in the usual way. It is, however, not necessary to carry out this preliminary operation. Consider the gnomonic zone-line Kq (of Fig.

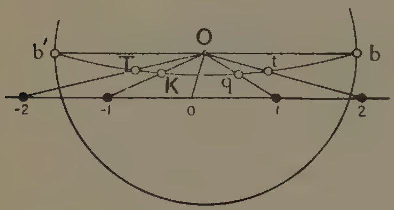


FIG. 47.

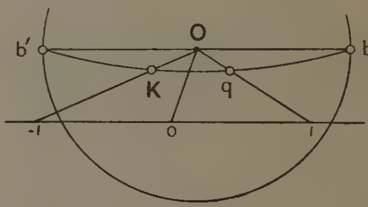


FIG. 48.

43) now reproduced as Fig. 47. The gnomonic poles are derivable from the stereographic poles by drawing radii from the centre and laying off twice the respective angular distances. But they all lie on a line parallel to the diameter $b'Ob$, and it is evident that if we were to translate this line to or from the diameter, the equal intercepts would remain equal, the scale alone being changed. The line would cease to be the gnomonic zone-line for a radius of 5 cms.; but it would still be a gnomonic zone-line for some other unspecified radius. If, then, the form development of copper sulphate were chiefly concentrated in the zone bq, it would not be necessary to proceed *formally* to the gnomonic projection in order to determine the indices. Any line, parallel to the diameter $b'Ob$ might be drawn (as in Fig. 48); then radii passing through the stereographic poles would intercept the line in regular steps and the indices would be read off. The gnomonic projection would really have been used, without having been formally prepared by the help of the main scale '*A*' of the protractor.

Conclusions.—The application of the 'stereographic method' to inclined zones can only be recommended with such zones as *ac* and *bc*, and also such simple zones as are typified by *cm*. In such cases it is extremely useful. If the zone has a more general position, it is better to proceed formally by the gnomonic method. Even the gnomonic method fails occasionally (in those rare cases where the indices are very complex), and recourse must then be made to logarithmic methods of computation. Experience would seem to show that such occasions never arise in ordinary practice, for out of some ten thousand laboratory substances, which have so far been examined, there are only a score or two which exhibit faces of complex indices demanding a logarithmic treatment.

Note on the Accuracy of Graphical Methods.—The accuracy of a graphical method depends on four main factors: (1) the nature of the problem; (2) the method of solving it; (3) the degree of perfection of the mechanical aids, as also their scale; and (4) the skill, or possibly more correctly the patience, of the worker. With regard to the stereographic projection, it was first shown by Penfield (*Amer. J. Sci.* 1901, Ser. IV, **11**, 25) that astonishingly accurate results are obtainable. Using his various protractors of 7 cms. radius, and a stand-lens to ensure accuracy of plotting, he found a greatest error of 6′ and an average error of 3′ in graphical solutions of two right-angled spherical triangles. In the solution of five oblique-angled spherical triangles the greatest error was 15′, the average 7′. Again, in a later paper (*ibid.*, 1902, **14**, 280) in which he subjected his methods to the most severe crystallographic test possible (that of an anorthic crystal), he found a maximum error of 25′, average error of 11′—an error greater than $\frac{1}{4}^{\circ}$ occurring in only three zonal angles out of 16.

A somewhat higher degree of accuracy is to be expected of the gnomonic projection. Using the metal protractor mentioned in the footnote of p. 2, Fedorov obtained the following results (*Annual of Russian Geol. and Min.* [Russian], 1905, **8**, 26; abstract in *Zeitsch. Kryst. Min.*, 1908, **44**, 89). The elementary gnomonogram of a cubic crystal was reduplicated and the rho-values of a series of eight gnomonic poles determined. In one case the error was 0′; in four cases 1′ each; in two cases 2′ each; in the last 4′. These, of course, are simple operations, but the more general of gnomonic determinations can be carried out with an average error of 3′ by the help of an ingenious (but costly) drawing machine, described in Fedorov's *New Geometry* [Russian], 1907, p. 108.

Results of the above refined degree of accuracy, transcending the limits within which the average crystal obeys the law of simple multiple intercepts, are not to be obtained without great care and patience. Possibly of greater practical importance is the question of the degree of accuracy obtainable with ordinary care and at a greater rate of execution. Such results, obtained by the use of the crystallographic protractor of 5 cms. radius, reveal an average error of $\frac{1}{4}^{\circ}$, and a maximum error of $\frac{1}{2}^{\circ}$, except under very unfavourable circumstances.

Note on the Frequency of Complex Faces on Minerals.—Recent experimental and observational work, rendered possible by the invention of the two-circle goniometer, bids fair to throw light on the not infrequent occurrence of complex faces on minerals. The work in question is that of D. N. Artemiev (*J. School of Mines, Petrograd* [Russian], 1908, **1**, 83, 165, 399; 1910, **2**, 252; *Zeitsch. Kryst. Min.*, 1911, **48**, 417) who conceived the happy idea of allowing a crystal, previously ground to the form of an hemisphere, to grow in a supersaturated solution. A large number of facets with relatively complex indices are first formed and these disappear as growth proceeds, so that the crystal finally acquires the usual simple form of the substance. Artemiev's work, combined with that of V. Goldschmidt and F. E. Wright (*Jahrb. Min. Beil.—Bd.* 1903, **17**, 355; **18**, 335) on the dissolution process, immediately suggests that the main mechanism, underlying the development of complex faces, is an alternating process of dissolution and growth, the dissolution producing rounded edges and corners, and the growth being responsible for a series of 'transitory' faces with complex indices. In the light of this interpretation (which of course requires confirmation by subjecting a crystal of natural shape to relatively mild solution-agencies, and subsequently allowing it to grow) the question whether or no a given crystal shall exhibit complex faces will simply depend on the circumstance whether

the various periods of growth have been sufficiently prolonged to efface completely the effects of intervening periods of dissolution. But this interpretation, it must be noted, cannot be expected to answer in all cases, for a few laboratory substances are known which develop curved surfaces under unquestionably continuous conditions of growth. An example ready to hand is that of the orthorhombic cæsium bitartrate, which according to the observations of J. P. Cooke (Groth's *Chem. Kryst.*, **3**, 320) exhibits a mixed development of plane and curved boundary tracts, an interesting confirmation of which has been recently furnished by the present writer's observation that the curved tracts are enantiomorphously developed on crystals of the corresponding *lævo*-bitartrate. Then, again, a few substances are known, which when crystallised under laboratory control exhibit isolated but brilliantly reflecting facets of high indices—the 'enigmatic facets' of Artemiev (*ibid.* [Russian], 1910, **2**, 275). Such indices as (31.7.11), for example, even if they be held to represent a vicinal replacement of (923) or (823) are remarkable in the case of an orthorhombic crystal which otherwise obeys every canon of simplicity in its form development—as indicated by the forms (010), (011), (110) and (101). But, whilst suggesting interesting themes for research, the occasional occurrence of these various complex types of development should not be allowed to obscure the general truth that crystals exhibit a remarkably simple type of form development, which ruler, compasses and protractor are quite competent to explore.

EXERCISES.

Exercise 9.—In continuation of Exercise 3 proceed from the stereographic to the gnomonic projection and determine the indices of both terminal and vertical faces by the method detailed in the text. If difficulties are encountered, leave aside and return to the problem after trying the following pair of exercises.

Exercise 10.—Prepare the primitive gnomonogram of the orthorhombic topaz, given the two angles (001) : (011) = 43°39′, (010) : (110) = 62°8′. [NOTE.—The whole point of this exercise lies in the use of a vertical face.] Further, prepare the primitive gnomonogram from the two angles (001) : (101) = 61° 1′, (010) : (110) = 62° 8′ . . .. The correctness of this work can be tested by comparing the dimensions of the gnomonograms with that of Exercise 6.

Exercise 11.—Prepare the primitive gnomonogram (with symmetry plane vertical) of the monoclinic iron vitriol from the angles β = 104°16′, (001) : (101) = 43°44′, (010) : (110) = 41°6′, and test its correctness when completed by means of the result of Exercise 7.

Exercise 12.—The fundamental angles of a certain vertical zone are *bm* (010) : (110) = 58°4′; *ba* = 87°6′. Determine by the stereographic method the indices of five other vertical faces making the following angles with *b* (measured clockwise, *i.e.* with the phi-values): 29° 29′, 39°54′, 117°33′, 137°54′, 149°2′.

Exercise 13.—The fundamental angles of a certain vertical zone are *bm* (010 : 110) = 60°16′, *bM* (010 : 1$\bar{1}$0) = 135°24′. Determine by the stereographic method the indices of three other vertical faces, with the phi-values: 102°30′, 144°40′, 151°23′.

Exercise 14.—The fundamental angles of a certain vertical zone are *bm* (010 : 110) = 53°3′, *bl* (010 : 210) = 64°48′. Determine by the stereo-

graphic method the indices of four other vertical faces, making the following angles with b (measured clockwise): 79°19′, 95°14′, 110°33′, 133°11′.

Exercise 15.—The fundamental angles of a certain vertical zone are bm (010 : 110) = 43°55′, ba (010 : 100) = 92°21′. Determine graphically the angular distances from the pole b(010) of the faces, n(120), $L(2\bar{1}0)$, $G(3\bar{2}0)$, $M(1\bar{1}0)$.

**Exercise* 16.—In a certain anorthic crystal with β = 111°27′ and bq(010 : 011) = 29°25′, bc(010 : 001) = 86°29. Construct the stereographic projection of this zone; plot in the face poles p and Q from the angles bp = 47°28′, bQ = 148°47′, and determine by the stereographic method the indices of p and Q.

Exercise 17.—In a certain monoclinic crystal (projected with symmetry plane vertical) the fundamental angles in the zone ac[100 : 001] are ac = 69°47′, ar(100 : 101) = 41°57′. Determine by the gnomonic method (why 'gnomonic'?) the indices of four other faces l, R, s, t, given the angles: al = 28°19′, aR = 110°36′, as = 138°14′, at = 151°47′.

Exercise 18.—Given a monoclinic crystal with the angular values ar(100 : 101) = 49°50′, ac(100 : 001) = 74°9, determine gnomonically the angle aR(100 : $\bar{1}$01).

> NOTE.—Unless the faces concerned are all actually developed, this simple problem is always involved in the derivation of the Fedorov complex-symbol, and is therefore of great importance in the identification of some 45 per cent. of all crystalline substances.

Problems similar to those of Exercises 12—18, but involving rectangular zones are not given here, as the exact solution (*i.e.* to the nearest minute) is obtainable by the help of the multiple tangent table (Chapter VI) in less time than the approximate solution by any graphical or instrumental method.

CHAPTER V.

CRYSTAL DRAWINGS.

'It is to a great extent the use of correct figures which has given Haüy's crystallographic method that great superiority which it has always enjoyed over the Wernerian method, both in accuracy and elegance; and to the study of Haüy's plates, far more than to the study of his writings, we must look as the point from which the subsequent labours of crystallographers started. . . . In the art of drawing in perspective, this method is called the Orthographic Projection, on account of the right angle, which the visual ray includes with the plane upon which the solid is represented. Herein it differs from the method followed by several modern authors, in which it is supposed that the eye of the observer is at once in two different places; but it agrees with the method employed in the works of Haüy.' W. HAIDINGER (1825).

'Unter diesem Titel [Ueber Krystallzeichnen] hat Herr V. Goldschmidt eine Notiz über ein sehr sinnreiches Verfahren zur Ausführung eines Krystallbildes in schiefer Projection veröffentlicht. Zu diesem Zwecke hat Derselbe die sehr rasch und bequem herzustellende gnomonische Projection zu Grunde gelegt. Sein Verfahren besitzt entschieden Vortheil vor dem üblichen Verfahren des axialen Zeichnens.'
E. S. FEDOROV (1898).

This section has been placed here in order to emphasise the fact that the preparation of a crystal-drawing no longer involves calculations.

To the beginner the exercise of the classical 'axial-cross method' of preparing a crystal drawing is very instructive, for it enforces the real significance of indices in a way which is possibly only exceeded by a subsequent study of the gnomonic projection. The method will also probably always remain the most time-saving in the preparation of such simple drawings as consist essentially of a pyramid-prism or bipyramid-prism of the more symmetrical systems, but to the research worker (who is necessarily almost wholly concerned with more general combinations of forms of the orthorhombic, monoclinic and anorthic systems) it should perhaps only have historical interest, for it is not so efficient as one, originally derived by V. Goldschmidt (*Zeitsch. Kryst. Min.*, 1891, **22**, 42) from the gnomonic projection, which has since been adapted to other forms of projection.[11] A horizontal plan of the crystal

[11] The method was adapted to the linear projection by E. S. Fedorov (*Zeitsch. Kryst. Min.*, 1898, **30**, 9) and to the stereographic projection by F. Stöber (*Bull. soc. franç. Min.*, 1899, **22**, 42) [possibly, first by B. Hecht, *Anleitung zur Krystallberechnung*, Leipzig, 1893 (as judged from an allusion made by E. Sommerfeldt, *Zeits. Kryst. Min.*, 1906, **41**, 164)]. Independent claims are also made (1) by G. Wulff (*Zeitsch. Kryst. Min.* 1902, **36**, 16-17) and (2) by C. M. Viola (*Grundzüge der Krystallographie*, Leipzig, 1904, 29, 46). Both gnomonic and stereographic projection methods (the latter from Viola) became known to S. L. Penfield, who introduced them to the notice of American workers without any essential modifications, in a paper (*Amer. J. Sci.*, 1906, **21**, 206) which can be regarded as a supplement to a well-known series, devoted to his own contributions to graphical methods (*ibid.*, 1901, ser. 4, **11**, 1, 115; 1902, **13**, 245, 347; **14**, 249; 1905, **19**, 39). The last-mentioned paper deals with crystal drawings, its essential contents being (1) a description of a special protractor for obtaining the axial cross (in any system) and (2) a method of drawing which is based on a transformation of the stereographic into the linear ('Quenstedt') projection, described briefly on p. 87. The use of the gnomonic projection in drawing has also been described by G. F. H. Smith (*Min. Mag.*, 1903, **13**, 309). A lucid account of the stereographic and gnomonic methods is included in H. E. Boeke's tracts (in German) on the corresponding projections (Berlin, 1911 and 1913 respectively). The gnomonic method has also been the

is first drawn. In the case of the cube, this is simply a square; but more generally, the preparation of the plan naturally involves appropriate truncations, which, however, can always be faithfully effected *in plan* with comparative ease (a most important point). The subsequent derivation from this plan of the customary parallel-perspective drawing (which to be precisely correct is an orthographic drawing in the style of Haidinger and Mohs, as opposed to the 'clinographic' drawing advocated by Naumann), from any desirable point of view whatsoever, is a purely mechanical process, which ensures the absolute fidelity of the drawing to the plan, and therefore to the crystal. Moreover, a cursory glance at the gnomonic (or stereographic or linear) projection is sufficient to reveal the most appropriate point of view for any given crystal. Accordingly there is no danger of wasting time in drawing a crystal from a point of view which subsequently proves to be ill-chosen.

Although the original Goldschmidt method is preferable to the linear adaptation (because it is, perhaps, as well not to attempt to master too many projections) and superior to the stereographic (because it is necessarily slightly more accurate), it seems expedient to commence with the stereographic variant as coming more natural to single-circle workers.

I. DRAWING FROM THE STEREOGRAPHIC PROJECTION.

1. **Preparation of the Plan.**—The application of the method can be illustrated in terms of a simple crystal of orthoclase of the habit shown in the final drawing Fig. 51, and exhibiting the forms $b(010)$, $m(110)$, $c(001)$, and $x(\bar{1}01)$. From appropriate data (say $\beta = 116°3'$, $cx = 50°16'$, $bm = 59°24'$), a stereographic projection of the upper half of the crystal is first prepared, care being taken only to draw those zones which are actually represented as crystal edges (*cf.* Fig. 49).

subject of two recent papers in English: by C. Palache (*Amer. Mineralogist*, 1920, **5**, 96) on the underlying principles, and by Miss M. W. Porter (*ibid.*, 89) on the practical details.

It is, perhaps, quite unnecessary to add that Fedorov's paper on the linear method also contains the key to the stereographic and also a 'cyclographic' or 'gramma-stereographic' method (though possibly in language which is somewhat obscure to those who have not made a special study of the Russian master's modes of expression) in order to illustrate how general has been the attention to improvements in technique during the last 30—40 years. Something like a score of instances could be cited in which two or more workers have arrived at the same result, not necessarily simultaneously, but quite independently of each other. Such parallel activities are no doubt partly due to that general awakening, which was bound to succeed a relatively quiet period, devoted to an assimilation of results associated with the names of Miller, Pasteur, Bravais, Sohncke and Sorby; but the main cause is probably to be sought in the circumstance that improvements in technique (however valuable) are not so much of the nature of discoveries as of practical applications of well-known principles. But this is scarcely the case in the present instance, for Goldschmidt's method of deriving from the gnomonic projection the direction of a crystal-edge in a general parallel-perspective Figure is so simple, that it amounts to a discovery.

The horizontal plan is drawn from the projection in accordance with the well-known truth that the direction in plan of any zone edge, bcb', is given by the normal to the diameter bOb'. The outline of the plan of our particular crystal is accordingly given by normals to the diameters Ob, Om, Om'; the relative lengths of the traces of the vertical forms $b(010)$ and $m(110)$ must be judged from their relative sizes on the crystal, and must also be taken in accordance with considerations of symmetry—unless the drawing is to represent a distorted crystal. Similarly, the direction in plan of the edge cx is given by the normal to the diameter aOa'. It will be readily understood that the exact position of the line cx in the plan is decided as in previous cases by the relative sizes of the forms $c(001)$ and $x(\bar{1}01)$ on the crystals; but it must not be overlooked that, as we are no longer dealing with vertical faces, we must also take into account the relative inclinations to the horizontal of the faces c and x. When viewed in plan, an inclined face is foreshortened to an extent which depends on the angle of slope; accordingly an excessively inclined face should be designedly drawn in plan quite narrow; otherwise the plan ceases to be a true representation of the crystal. In the particular case in question, the faces $c(001)$ and $x(\bar{1}01)$ happen to be about equally inclined, so that the 'inclination-factor' (equal to the cosine of the angle of slope) can be neglected: the line cx is drawn so as to give the face $c(001)$ about twice the area of the face $x(\bar{1}01)$, in accordance with the actual crystal habit.

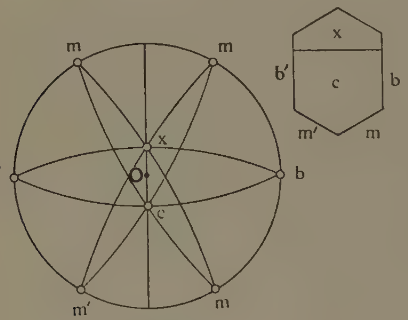


FIG. 49.

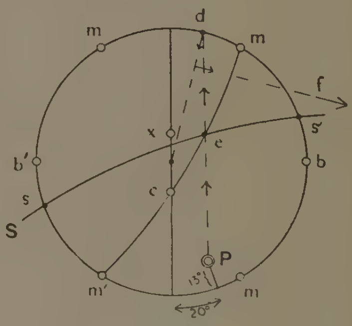


FIG. 50.

2. **Relation of the Final Drawing to the Plan.**—Now a horizontal plan is a picture of the crystal as viewed from a point above the centre O of the projection. If the point of view were normal to the face $m(110)$, the plan would be termed an 'elevation' on m. If, on the other hand, the point of view were represented by the pole P (*cf.* Fig. 50) fairly close to the pole m, the 'plan' would be generally similar to the figures of orthoclase given in books on mineralogy—the main differ-

ence being that the conventional drawings are made from a slightly different point P, namely, one derivable from that of $a(100)$ by a rotation of $18\frac{1}{2}°$ and an approach towards the centre of $9\frac{1}{2}°$, instead of the values of $20°$ and $13°$ adopted here (the reason for this new choice will soon become evident).

The question now arises: how can one obtain this new plan? It will be clear that if by some means or other we can transform (turn over) the whole of the projection so as to bring the pole P (*i.e.* the eye) to the centre and determine the new point (d), in which a zone cm' after transformation cuts the primitive, then a line drawn normal to the new diameter Od will represent the edge cm in the new plan (*i.e* in the finished drawing). The transformation can be easily effected. The only auxiliary required is a meridian circle S, of which P is the pole. This meridian circle can be easily drawn by the help of the crystallographic protractor, provided the pole P is not at a greater angular distance from the centre than $77°$. The centre of the $77°$ meridian is given by the last graduation ($154°$) of the main scale. The accurate drawing of a flatter meridian (say of the standard $9\frac{1}{2}°$ inclination) involves the use of a flexible ruler. With the help of this auxiliary circle S we are in a position to determine any edge direction in the new plan: in particular, the new edge cm'. Note the point e in which the zone cm' intersects S. From P project the point e to the primitive. The point d so obtained represents the new point in which the zone $m'c$, after transformation, cuts the primitive; and the normal f to the diameter Od is the edge-direction required. The above construction is quite general. In particular it holds for the vertical zone, but as this actually intersects S in diametral points (s and s'), any subsequent projection from P to the primitive is unnecessary. The normal to the diameter sOs' accordingly represents the vertical direction in the new plan or crystal drawing.

It is perhaps worth while pointing out that the derivation of any new zone-edge such as f can be *rapidly* effected as follows. With the fine point of a hard pencil locate the point e. Bring up the edge of a set square (with the left hand) till it rests against the pencil held firmly at e, and rotate it round this support till the edge passes through P. Now shift the pencil point from e to d, and rotate the set-square anew about d as a support till the edge passes through O. The normal to Od is then taken off in the usual way by the aid of a second (right angled) set-square.

The only question that remains refers to the *lengths* of the various edges in the new plan. These lengths have not to be determined afresh: no matter from what new point of view the crystal is depicted, the lengths are implicitly determined by the original horizontal plan. From the various points of the original plan, lines are drawn parallel

to the diameter *OP*. The lines are interrupted in Fig. 51, in order to prevent confusion. A reader might do well at this stage to pencil (*not* ink) in the interrupted parts between the upper and lower sets of arrows, and also continue the line *f* in pencil across the drawing. The direction of the new edge *cm′* is given by *f*: the *length* of the edge is determined by the two points in which *f* intercepts the relevant parallels drawn down from the plan; and so on with every other edge. The only lengths not determined by the original plan are the vertical edges: one of these has of course to be defined arbitrarily as to length, in accordance with the general habit of the crystal.

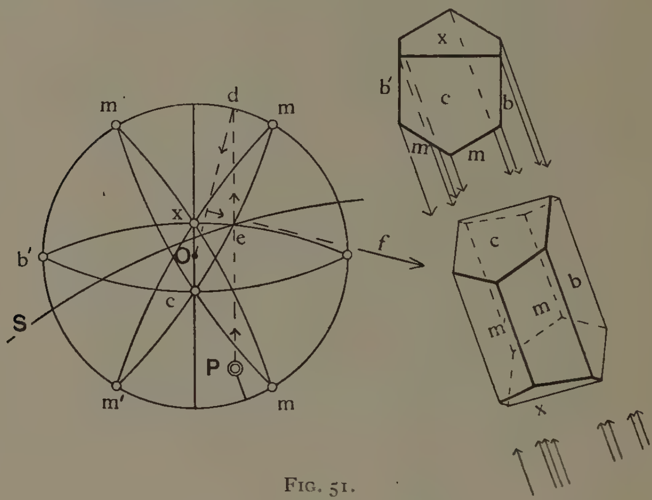


FIG. 51.

3. **Practical Details.**—A few practical details may now be useful. A commencement is made on the upper part of the crystal (both back and front), and it is best to begin with a large, front face, like *c*(001). The various peripheral edges of this face are derived in turn, by the method fully described for the edge *cm′*. The actual lengths of the edges are determined by the system of parallels drawn down from the horizontal plan, and if the work is carried out carefully, the last peripheral edge will pass through the terminal of the first. Then an adjacent upper face is selected; its outline worked out—and so on till the whole of the upper terminal faces have been drawn. In the case in point, there is only a second upper face, namely, *x*($\bar{1}01$), so that the upper termination can be drawn in anything under ten or fifteen minutes. The problem as to which edges lie at the back of the crystal (as viewed in the position adopted) is automatically solved by the projection. The two poles *m′* and *c* both lie in front of the arc *S*: the edge in question is accordingly in front. Of the two poles *c* and *x*, the latter lies behind the arc *S*: the edge in question is a

boundary-edge of the drawing. The two poles $x(\bar{1}01)$ and $b'(0\bar{1}0)$ both lie behind the arc S: the edge in question lies at the back of the crystal.

So much for the top of the crystal. The vertical edges have in reality already been drawn, for they are a part (not necessarily the whole) of the parallel lines drawn down from the horizontal plan. Care must be taken to reject any such parallels as are not actual edges. Thus in Fig. 51 there are three parallel lines coming down from the trace (in plan) of the face $b(010)$. Of these three lines only the two outer lines must be preserved as vertical edges, for the middle line is drawn from a 'non-angular,' peripheral point (in the plan), which therefore does not represent a vertical edge, its office being to indicate the point in which the edges bc, bx and cx meet.

There remains the lower termination of the crystal. If, as is generally the case, the crystal exhibits a centre of symmetry, the following procedure may be adopted. An appropriate point is selected on one of the vertical edges, and one of the lower faces is gradually mapped out, by drawing lines parallel, *but in the reverse direction,* to the outline of the equivalent upper face, the length of each line being carefully checked or actually determined by the application of a pair of dividers (point-compasses) to the upper edge. In the rare cases in which the lower termination is dissimilar, the upper termination and vertical faces are drawn as before; then the internal details of the original plan are rubbed out and substituted by a plan of the lower termination, *as viewed from above.* The necessary new parallels are drawn down, and the lower termination worked in as before from the stereographic projection. It need scarcely be added that there is no need in the projection to operate with poles which lie on the lower hemisphere, for they can be replaced (for the present purpose) by the poles of parallel faces lying on the upper hemisphere.

When the drawing has been completed, it can be pricked through on to another sheet of paper (so arranged that the crystal adopts an upright posture) and inked in. Some authors reproduce the horizontal plan in addition to the final drawing. The two pictures certainly supplement each other admirably.

The treatment of twin-crystals presents no unusual difficulties. Unfortunately, space will not allow of any detailed description.

4. **Advantages of the Method.**—The more salient advantages may now be enumerated :—

(1) If a flexible ruler be available (or some other device such as beam compasses, admitting of the accurate drawing of a flatter arc than one of 13° inclination), the crystal can be drawn from any desired point of view whatsoever. This is a matter of some importance, for

it occasionally happens that the customary point of view is not suitable. If one of the face-poles lies on (or near) the arc *S*, the face in question becomes foreshortened (or nearly so) into a line, in the finished drawing. This defect can be remedied at the outset, by a slightly different selection of the arc *S* and its corresponding pole *P*. (Parenthetically, it may here be mentioned that if the pole *P* be selected so as to be coincident wih a face-pole, then the finished drawing is really a plan on that face.)

(2) The method avoids the preparation of an axial-cross, and therefore in the monoclinic system and especially in the anorthic system, a considerable amount of preliminary work is avoided.

(3) The method of deducing the direction of each edge is much simpler. With the axial-cross method the indices of the two faces have to be taken into consideration before the direction of the common edge can be found.

(4) The last, and not least, advantage is that the fidelity of the final drawing can be relied upon. It is comparatively easy to draw the initial horizontal plan true to the crystal, and the final drawing is necessarily correspondingly true. With the axial-cross method, as usually practised, the drawing is prepared from a primitive figure (without the aid of a horizontal plan) by a series of truncations. Much judgment is needed in making these successive truncations. In the case of an anorthic crystal, the probability appears to be small that the final result is an accurate picture of the crystal.

5. **Disadvantages of the Method.**—The only disadvantage is that arcs are sometimes difficult to draw accurately. This should not be taken to imply that the method is not to be recommended—in support of which the present writer would add that he used the method for two years, without the help of a special protractor, with satisfactory results.

EXERCISES.

I.—*Drawing from the Stereographic Projection.*

Exercise 19.—Prepare a horizontal plan and general parallel-perspective figure of orthoclase, as described in detail in the first part of this chapter.

Exercise 20.—Draw a horizontal plan and general perspective of a cube from any suitable position. How do you propose to determine the vertical dimension of the cube?

Exercise 21.—Draw a horizontal plan and general figure of a crystal of idocrase of a prismatic habit and exhibiting the forms *m*(110), *p*(111) and *c*(001). Given $cp = 37°13'$.

Exercise 22.—Draw a horizontal plan and general figure of a singly terminated crystal of topaz of a prismatic habit and also exhibiting $c(001)$, u (112) and $o(111)$ about equally developed. Given $mm(110 : 1\bar{1}0) = 55°43'$, $cu(001 : 112) = 45°35'$, $co(001 : 111) = 63°54'$.

Exercise 23.—Draw a regular pentagonal prism as an exercise in a non-crystallographic figure (or, alternatively, a regular hexagonal prism).

**Exercise* 24.—Draw the regular tetrahedron (111).

**Exercise* 25.—If a flexible arc-ruler be available, draw a horizontal plan and general figure of a crystal of albite (of the pericline habit illustrated in Dana's *System*, p. 328, Fig. 3) exhibiting the forms $b(010)$, $m(110)$, $M(1\bar{1}0)$, $c(001)$ and $x(\bar{1}01)$. The projection is best made from the two-circle data: (1) phi-values, $b(0°0')$, $m(60°30')$, $a(90°15')$, $M(119°52')$, $c(81°51')$, $x(279°6')$; (2) rho-values, $c(27°\ 1')$, $x(64°12')$.

II.—*Drawing from the Gnomonic Projection.*

Exercises 26—28.—Repeat Exercises 20—22, but with the adoption of the 'standard orientation.'

Exercise 29.—Prepare a horizontal plan and general figure of orthoclase, as described in detail in the second part of this chapter.

Exercise 30.—Given $a : b : c := 1{\cdot}000 : 1 : 1{\cdot}456$, prepare a gnomonic projection of this pseudo-tetragonal orthorhombic crystal exhibiting the forms (110, (102) and (011), and discuss the suitability of the standard position for drawing purposes.

Exercise 31.—Draw the albite crystal from the data already given under Exercise 25 above; or, alternatively, draw the plan and general figure of the anorthic bibromo-inosite tetracetate (reproduced as Figs. 80—81, p. 110) from the data there given (preferably from the Fedorov angular elements of the crystal).

II. DRAWING FROM THE GNOMONIC PROJECTION.

Although the stereographic adaptation is good, the original gnomonic method is better. Straight lines can be drawn more accurately than circles; moreover, there is no necessity to invoke the aid of a flexible ruler or beam-compasses. In particular, no difficulty arises in the preparation of a drawing from the arbitrarily selected 'standard position.' It will be instructive to prepare a drawing of the same crystal of orthoclase, from the standard point of view, involving a rotation of $18\frac{1}{2}°$ and a tilt of $9\frac{1}{2}°$.

The gnomonic projection may be prepared from the stereographic projection in the usual way for single-circle readings (the stereographic

poles of the terminal faces are not inserted in Fig. 52). The gnomonic poles $c(001)$ and $x(\bar{1}01)$ represent one side of the elementary gnomonogram. The other side is obtained by making use of the vertical faces. Lines are drawn through the gnomonic poles of c and x parallel to the radius Ob, and a radius Om (or Om') is translated until it passes exactly through the gnomonic pole of c; its intersection with the line drawn through x parallel to Ob, gives the other side of the gnomonogram.

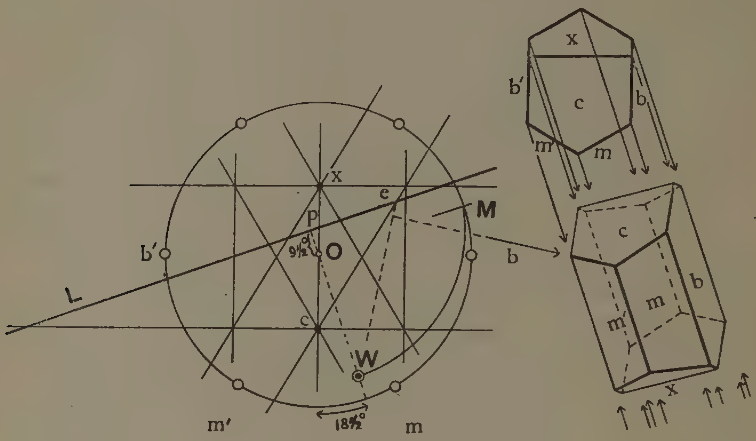


FIG. 52.

In Fig. 52 this primitive gnomonogram is duplicated, and both are divided diagonally into four sectors. All these diagonal lines (zone-lines) are necessary to the preparation of the drawing, for they represent actual edges; otherwise they would not have been drawn.

1. **The Guide-Line and the Angle-Point.**—Corresponding to the arc S and its pole P (of Figs. 50—51), we have here to deal with a 'guide-line,' L (German, 'Leitlinie'), and the 'angle-point,' W (German, 'Winkelpunkt') of Fig. 52. The guide-line is the trace of the plane on which the crystal is ultimately projected. Its position is determined by drawing a radius pO at an angle of $18\frac{1}{2}°$ with the front-back direction cx. The main scale of the protractor is placed along this radius and the point p, having an angular distance of $9\frac{1}{2}°$ gnomonically (or $19°$ stereographically) is located. The perpendicular at p, to the radius pO is the guide-line required. Its angle-point, W, is drawn: either (1) by drawing the radius OM parallel to the guide-line, and striking an arc with centres p and radius pM so as to intercept pO; or (2) by laying off W directly by the help of the main edge of the protractor, for W has stereographically an angular distance of $(90°—9\frac{1}{2}°)$ from the centre.

2. **The Plan and the Finished Drawing.**—The preparation of the final drawing is modelled on lines already described for the stereographic method. A horizontal plan is first prepared; a system of parallels brought down from the various points of this plan (a reader would do well, as previously, to pencil in these interrupted parallels); the directions of the edges found by a simple method; the bottom of the crystal copied from the top by inversion, and so on.

The direction of a crystal edge in horizontal plan *is simply the normal to the corresponding zone-line.* It will be well to consider the methods of drawing such zone-lines in general. If two terminal faces are given, the zone-line is simply obtained by joining their gnomonic poles. If, on the other hand, one of the faces is vertical (as, for example, in the case *cm*), the line can be constructed in two ways: (1) the simplest way is to translate the radius *Om* till it passes through the gnomonic pole of *c*; (2) the second way is to deduce its direction from the indices. If we add the indices of *c*(001) and *m*(110), we get the indices of a possible face (111) in the zone *cm*. The same is true if we subtract the indices of one from the other, *e.g.* ($\bar{1}\bar{1}1$). Either of these faces (whichever is the more convenient) can be incorporated into the gnomonic projection by means of the plane co-ordinates (11) and ($\bar{1}\bar{1}$) respectively, and the zone-line drawn, joining this new pole and the pole of *c*.

The horizontal plan of Fig. 52 is obtained by drawing lines perpendicular to the corresponding zone-lines in the gnomonic projection. It is necessarily an exact replica of the correponding plan of Fig. 51. The difference begins with the system of parallels, for these must now be drawn at an angle of $18\frac{1}{2}°$ to *cx* (instead of 20°).

The obtainment of the direction of a zone-edge in the final drawing is more direct than with the stereographic method. The gnomonic zone *m′c* cuts the guide-line *L* in the point *e*. A set-square is placed so that an edge is in alignment with *e* and the angle-point *W*; the normal thereto is the direction required.

It occasionally happens that the intersection of zone-line and guide-line lies at an inconveniently great distance from the centre of the projection. In this case, an *ad hoc* reduction of scale may be effected. This reduction is brought about by translating zone-line, guide-line and angle-point proportionately towards the centre (the central distances of zone-line and guide-line being of course measured along perpendiculars passing through the centre). An alternative and, perhaps, better procedure is to revert to the stereographic method for the particular edge in question—for the point *W* is simultaneously the pole *P* of the $9\frac{1}{2}°$-meridian required, and the latter presents no difficulty if a flexible ruler be available.

3. **Advantages of the Method.**—The method has naturally all the advantages of its stereographic adaptation. These need scarcely be enumerated. It may, however, be as well to emphasise that if the guide-line L of the standard position passes too near the gnomonic pole of any face, a new and more appropriate guide-line (and angle-point) can be used.

4. **Final Comparison.**—A comparison of the final drawings from the two methods is instructive. The effect of the difference in tilt on the apparent size of the basal plane $c(001)$ is only just appreciable, and it follows that drawings effected from the stereographic projection with a tilt of $13°$ are 'practically' interchangeable with those drawn from the standard position. In the rectangular systems the advantage, such as it is, would seem to lie with the $13°$ tilt. There is of course no intrinsic virtue in the $9\frac{1}{2}°$ tilt, for the tilt should vary with the requirements of the individual case.

In conclusion, it may be mentioned that an axial-cross can be constructed from either projection, and a horizontal plan from the axial-cross; but it does not seem necessary to outline such circuitous methods here, although they do not appear to have been previously described.

Supplementary Note (added in proof-correction).—Both clinographic and orthographic methods of drawing *from an axial cross* are the subject of a geometrically authoritative paper by L. Burmester (*Zeitsch. Krist.*, 1922, **57**, 1) from which it appears: (1) that Haüy's method of drawing, though practically indistinguishable in its results from the orthographic method, is in principle a clino-graphic method; (2) that the orthographic method was really devised by Haidinger (an acknowledgment of which is to be found in Mohs' writings); and (3) that Naumann (who contributed more than anyone else to the development of the clinographic variant by prescribing methods of deriving the axial cross in the monoclinic and anorthic systems) eventually became persuaded, as a result of discussions with G. Rose, that Haidinger's orthographic method is better adapted to crystallographic purposes. This conclusion, it must be noted, is not shared by Burmester, who has been an advocate of the clinographic method for more than 50 years in the belief that it is simpler and more correct in principle. As there would here appear to be a danger of a confusion of the main issue, the present writer would express the view that although Burmester's paper is profound in many geometrical respects, yet as a whole it only touches the fringe of the drawing problem of the crystallographer, who is not so much concerned with the relative merits of the clinographic and orthographic axial crosses, as: (1) with ways and means of avoiding the considerable labour involved, both in the preparation of any cross and in its application to the preparation of the drawing (the transformation of indices into intercepts); (2) with the possibility that the position may be unsuitable; and (3) with the difficulty felt by anyone who is not a professional geometrician (and therefore to a large extent dependent on hard and fast rules) in re-drawing an axial cross suitable to the occasion. Apparently the first crystallographer on record to offer a radical solution (by the excision of the axial cross itself) was Goldschmidt, whose first attempts (*Ueber Projection und graphische Krystallberechnung*, 1887, pp. 82 and 90) were not particularly successful, but whose elegant (and, to the present writer's mind, final) solution—the basis of the present Chapter—was not long in forthcoming. Less drastic methods (methods of cure rather than of prevention) were subsequently attempted by Penfield (*loc. cit.*) and by Travis (*Zeitsch. Kryst. Min.*, 1910, **47**, 600) who were unaware that the problem had already been solved from a more general standpoint. Both contributions took the form of special protractors—Penfield's facilitating the preparation of the cross; Travis' leading to the determination of the edge-directions.

CHAPTER VI.

CRYSTALLOGRAPHIC CALCULATIONS.

'The Crystallographic Notation adopted in the following Treatise is taken, with a few unimportant alterations, from Professor Whewell's Memoir 'On a general method of calculating the angles of Crystals,' printed in the Transactions of the Royal Society for 1825. The method of indicating the positions of the faces of a Crystal by the points in which radii drawn perpendicular to the faces meet the surface of a sphere, was invented by Professor Neumann of Königsberg (Beiträge zur Krystallonomie) and afterwards, together with the notation, re-invented independently by Grassmann (Zur Krystallonomie und geometrischen Combinationslehre). The use of this method led to the substitution of spherical trigonometry for the processes of solid and analytical geometry in deducing expressions for determining the positions of the faces of crystals and the angles they make with each other. The expressions which in this Treatise have thus been obtained, are remarkable for their symmetry and simplicity, and are all adapted to logarithmic computation. They are, it is believed, for the most part new.' W. H. MILLER (1839).

'Le livre de Miller, traduit en français par Sénarmont, reste dans son élégante concision, comme un monument achevé de l'application, à la cristallographie, des méthodes de la géométrie.' E. MALLARD (1879).

Crystallographic calculations can be divided into two distinct parts: (1) the computation of some form of 'elements'; and (2) the computation of the 'theoretical' values of the angles measured, *i.e.* of values which are rigorously consistent with the 'elements.'

The subordination of a crystal to the law of simple multiple intercepts or simple indices makes it theoretically possible to utilise all the angles in calculating a set of elements. Methods of carrying this theoretical possibility into practice have been elaborated, amongst others, by J. Beckenkamp (*Zeitsch. Kryst. Min.*, 1894, **22**, 376) for single-circle measurements; and by V. Goldschmidt (*ibid.*, 1893, **21**, 221) and G. Wulff (*ibid.*, 1904, **38**, 1), for two-circle measurements.[13] Such methods, with the exception of Goldschmidt's method which has been adopted by many two-circle workers, have not found any measure of acceptance. It is much more usual to select the minimum number of fundamental angles demanded by the system. Such selections are somewhat arbitrary, as there is generally a greater number of trustworthy angles available than the minimum required; but the general

[13] Of the two-circle methods, that of Wulff is the more rigorous, whilst that of Goldschmidt is the more accessible to the general worker, because it is based on the simple linear properties of the gnomonic projection. The latter is generally practised in connexion with the zone-type of adjustment on the goniometer, as opposed to the face-type; and the resulting measurements of an anorthic or monoclinic crystal are, in general, neither the interfacial nor the interzonal angles of classical crystallography. This is not the place to discuss the relative merits of the zone- and face-adjustments, but it may be stated that the Goldschmidt method can be readily adapted by suitable changes of formulæ, to the face-adjustment method of goniometry: *i.e.* to the system of measurement in which one circle registers the interfacial angles of single-circle goniometry, whilst the other registers the equally important interzonal angles.

consensus of opinion is in favour of this custom, as satisfying all reasonable requirements of everyday work. Crystallographic calculations, then, may be generally taken to involve: (1) the computation of a set of elements from 'fundamental angles'; and (2) the subsequent computations of the remaining angles, on the assumption that the fundamentals are correct.

The methods generally adopted are trigonometrical. They comprise the solution of plane and spherical triangles and the special formulae, derived therefrom by W. H. Miller (*A Treatise on Crystallography*, 1839) connecting the indices and angular values of four tautozonal faces (or coplanar zone-axes). These Millerian formulae derive their extraordinarily practical importance from two remarkable features of crystal development: (1) the faces are generally included in a small number of zones, which (2) frequently intersect in a common face. A slightly exaggerated (and therefore not quite typical) example of these two features is exhibited by the anorthic copper sulphate. The eighteen forms, which have been so far observed, all lie in four zones passing through the face $b(010)$. Corresponding, simplifying properties are exhibited by crystals of other systems, in so far as the symmetry will allow.[14]

The general subject of calculations has been examined by numerous crystallographers. It is evident that any further improvements must refer to questions of detail. The following pages, appertaining exclusively to the Millerian formulae, will be devoted to some improvements in detail. The first main section deals with the Millerian formulae in general, the principal object being to emphasize both the greater efficiency of the cotangent form and the prevalence in crystallography of the simple 'harmonic case.' The second section deals with the special case in which two out of the four tautozonal faces (or coplanar axes) are inclined at 90°. This section is accompanied by a 'multiple tangent table,' which nearly wholly replaces logarithmic computations in systems ranging from the cubic to the orthorhombic, and also brings a certain amount of relief in the monoclinic system.

I. THE MILLERIAN FORMULAE.

It is important to note in what follows that the letters A, B, C, D represent face-poles (or zone-circles) in consecutive order; thus, $A—B—C—D$. In other words A and D are the terminals and B lies between A and C. In allocating these letters to any concrete problem,

[14] It is of course the first of these two features that makes single-circle goniometry at all practicable. If n boundary planes (parallel pairs of faces) of an anorthic crystal were so distributed that no three lay in a zone (whilst obeying a 'law of rational indices') the measurement of $2(n-2)+1$ 'zones' would be necessary; 12 planes, then, would involve 21 zones. If crystals were so developed, it is obvious that instead of working out his celebrated formulæ (published in 1839), Miller would have devised the two-circle goniometer some thirty-five years before he actually did!

it does not matter, of course, which terminal is taken to be A, and which D.

1. **The Direct Sine Formula.**—If we have the four tautozonal faces, $A(abc)$, $B(def)$, $C(hkl)$, $D(rst)$—or alternatively, four coplanar zone-axes, $A[abc]$, $B[def]$, $C[hkl]$, $D[rst]$[15]—then the following formula holds :—

$$\frac{\sin AB}{\sin AC}\cdot\frac{\sin DC}{\sin DB}=\frac{\begin{vmatrix}abc\\ \overline{def}\end{vmatrix}}{\begin{vmatrix}abc\\ \overline{hkl}\end{vmatrix}}\cdot\frac{\begin{vmatrix}rst\\ \overline{hkl}\end{vmatrix}}{\begin{vmatrix}rst\\ \overline{def}\end{vmatrix}}\text{ or, say, }\frac{\begin{vmatrix}A\\ \overline{B}\end{vmatrix}}{\begin{vmatrix}A\\ \overline{C}\end{vmatrix}}\cdot\frac{\begin{vmatrix}D\\ \overline{C}\end{vmatrix}}{\begin{vmatrix}D\\ \overline{B}\end{vmatrix}}\ \ldots\ldots(1)$$

where, for example, $\begin{vmatrix}abc\\ def\end{vmatrix}$ or, say, $\begin{vmatrix}A\\ \overline{B}\end{vmatrix}=ae\text{-}db$, or $bf\text{-}ec$ or $af\text{-}dc$ (due care being taken to make a uniform selection throughout, and also such a selection as does not lead to an indeterminate result, involving zero).

The above formula (which, in an abbreviated form, is engraved on the crystallographic protractor) is applicable to the case where *all* the angles are known, as also the indices of three out of the four poles. Its office is to find the indices of the fourth pole—which may be A, B, C or D indifferently. Consequently, it need have no application in these days of precise graphical methods, except in exceedingly rare cases (those in which the indices of the fourth pole are very complex).

2. **The Converse Cotangent Formula.**—Miller showed that the four sines of the above formula can be re-arranged into another form involving cotangents. Thus :—

$$\frac{\sin AB}{\sin AC}\cdot\frac{\sin DC}{\sin DB}=\frac{\cot AC-\cot AD}{\cot AB-\cot AD}=\frac{\begin{vmatrix}A\\ \overline{B}\end{vmatrix}}{\begin{vmatrix}A\\ \overline{C}\end{vmatrix}}\cdot\frac{\begin{vmatrix}D\\ \overline{C}\end{vmatrix}}{\begin{vmatrix}D\\ \overline{B}\end{vmatrix}}=\frac{p}{q}\ \ldots\ldots(2)$$

and this by a simple re-arrangement becomes :—

$$p\cot AB-q\cot AC=(p-q)\cot AD\ \ldots\ldots\ldots\ldots(3)$$

This third formula (also engraved on the crystallographic protractor) is applicable to the converse case, in which all the indices (and, therefore, the fraction p/q) and also two angles (say, 'fundamental angles') are known, and it is required to compute the third angle. A glance at the formula will show: (1) that it does not matter which of the three angles AB, AC or AD we have to compute—given any two we can compute the third without any drastic re-arrangement of the formula (a not unimportant point); (2) that the formula ultimately involves the addition or subtraction of *natural* cotangents (a five-figure table of which is included in this book), and, therefore, that care must be taken with algebraic signs, since $\cot x=-\cot(180^\circ-x)$.

15 In what follows this somewhat troublesome citation of the alternative case of zone-axes (instead of faces) will be omitted.

It is now convenient to discuss the cases in which the formula must be used with great circumspection. A theoretical angular value close to 90° should not be computed from two measured angles which are excessively small (**Note:** an angle close to 180° is small from the standpoint of the cotangent formula, for it has to be changed into its supplement). Suppose, for example, we have four poles $A(010)$, $B(110)$, $C(100)$, $D(1\bar{1}0)$, and that the true (unattainable) values of AB and AD are 20°0′ and 159°0′ respectively; then the true value of AC would be 85°56′. Now, measured angles can scarcely ever be 'true'; the error can be easily ± 3′. If the two *measured* angles were 19°57′ and 158°57′ respectively, the computed value of AC becomes 85°31′. The deviation from truth is 25′. If, on the other hand, we selected AB and AC as fundamentals, the greatest deviation of the computed value of AD from the 'true' value is 4′, and is therefore of the same order as the supposed original errors. The inference is that in any zone, *one of the angles selected as fundamental should be as close to 90° as* possible, whilst the other angle should not be excessively small. The first condition is italicised because it is the more important.

The Alternative Converse Formula.—Miller's second way out of the difficulty is less direct. It is presupposed that AB and BC are the known angles. Then

$$\frac{\sin AB}{\sin AC}\cdot\frac{\sin DC}{\sin DB}=\frac{p}{q} \text{ or } \frac{\sin DC}{\sin DB}=\frac{p}{q}\cdot\frac{\sin AC}{\sin AB}=\text{, say, } \tan\theta \ldots\ldots\ldots(4)$$

Then it can be proved that:

$$\tan\tfrac{1}{2}(DB+DC)=\tan\tfrac{1}{2}CB/\tan(45^\circ-\theta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

By successively evaluating equations (4) and (5) we obtain *half* the sum of DB and DC. And as *half* the difference is already known (being half the value of CB), the angles DB and DC are finally obtained by addition and subtraction.

The θ-formula appears to be more extensively used than the cotangent formula: possibly because tables of natural trigonometrical functions are not so generally accessible as those of logarithmic functions. In any case the formula suffers from the following disadvantages: (1) it takes much longer time to evaluate; and (2) in some cases the angle θ must be computed to seconds if the 'correct result' is to be obtained to the nearest minute. This disadvantage, be it observed, is peculiar to the θ-formula—it has no connection with the deviations from truth which may occur under specially unfavourable conditions. If the angle θ is always evaluated to seconds, the two formulae give the same results to the nearest minute.

3. **Examples of the Cotangent Formula.**—It may now be useful to give two worked-out examples of the cotangent formula. The second has been selected with a view to illustrating the necessity of exercising care in the algebraic signs, when one (or more) of the angles is greater than 90°. At this stage the value p/q may well be obtained by cross-multiplication, and not by an inspection of the gnomonic projection as described on p. 65.

Example 1.—Given in copper sulphate AB (010 : 120) = $x°$, AC (010 : 110) = 52°59′, AD (010 : 100) = 79°6′. To find the value of x. We evaluate by cross-multiplication the right-hand side of formula (2) and obtain $p/q = 1/2$; *i.e.* $p = 1$, $q = 2$. Putting these values into the general formula (3), namely, $p\cot AB - q\cot AC = (p-q)\cot AD$,

we obtain $\cot AB - 2 \cot AC = -\cot AD$. Since it is AB that is required, we re-arrange accordingly and obtain $\cot AB = 2 \cot AC - \cot AD = 2 \cot 52°59' - \cot 79°6' = 2 \times 0{\cdot}75401$ (*i.e.* $1{\cdot}50802$) $- 0{\cdot}19257 = 1{\cdot}31545 = \cot 37° 14\frac{1}{2}'$ (say, $37° 15'$).

Example 2.—Given in the same zone, AB (010 : 110) = 52°59′, AC (010 : 100) = 79°6′. Find AD (010 : 1$\bar{3}$0). Evaluating formula (2), as before, we obtain $p/q = 3/4$, *i.e.* $p = 3$, $q = 4$. Putting these values into the general equation (3), we obtain $3 \cot AB - 4 \cot AC = -\cot AD$. Now, it is AD that is wanted, and signs could be changed throughout to make it positive, but this is unnecessary. We have, then, $-\cot AD = 3 \cot 52°59' - 4 \cot 79°6' = 3 \times 0{\cdot}75401$ (*i.e.* $2{\cdot}26203$) $- 4 \times 0{\cdot}19257$ (*i.e.* $0{\cdot}77028$) $= 1{\cdot}49175 = \cot 33°50'$. Now, if $-\cot AD = \cot 33° 50'$, then $\cot AD = \cot 180° - 33° 50' = \cot 146° 10'$.

4. **The Harmonic Case.**—Four face-normals are harmonic when they exhibit the simple relation shown in Fig. 53, in which a line K, drawn parallel to one of the terminal rays (say, A), makes equal intercepts on the other three (*i.e.* $dc = cb$). Now, whenever four poles A, B, C, D are harmonic, the general Millerian formula acquires the simple form :

$$\cot AB + \cot AD = 2 \cot AC.$$

(This formula is easily committed to memory, if it be associated with an idea that the intermediate angle AC should be of the nature of an arithmetical mean of the smallest AB and the largest AD.) The harmonic case is particularly common (something of the order of 95—99 per cent. of actual cases are harmonic). It will therefore be realised that if these harmonic cases can be immediately recognised either as a result of experience or by some simple test, then it is no longer necessary to cross-multiply indices; one simply writes down the equation $\cot AB + \cot AD = 2 \cot AC$, and goes ahead with the evaluation. It is convenient to recognise three tests.

Test 1.—The first is really that of Fig. 53. It is an obvious test to apply in the vertical zone of an anorthic crystal (or of the zone ac

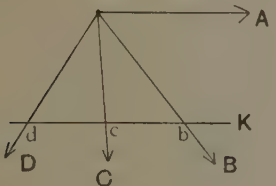


FIG. 53.

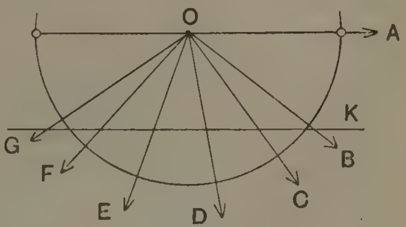


FIG. 54.

of a monoclinic crystal which has been projected on the plane of symmetry). In actual work the 'test' will probably have been already applied, namely, in the graphical determination of indices by the

'stereographic method' (p. 42). Fig. 54 represents the vertical zone of copper sulphate. The various face-normals are intersected by the line K drawn parallel to a terminal face-normal OA. A glance at such a diagram is sufficient to reveal the fact that all the following quartettes are harmonic: A, B, C, D; A, C, D, E; A, D, E, F; A, E, F, G; A, B, D, F; A, C, E, G. And it is nearly always possible to carry out a series of computations in such an order that the individual computations are harmonic. NOTE.—It is, perhaps, not correct to say that a glance at Fig. 54 is sufficient, for owing to an optical illusion the intercepts appear to be unequal (a measurement with point-compasses will show they are really equal).

Test 2.—The second test is especially applicable to cases which involve one vertical and three terminal faces. The test consists of an inspection of the gnomonic projection. Every zone in this projection

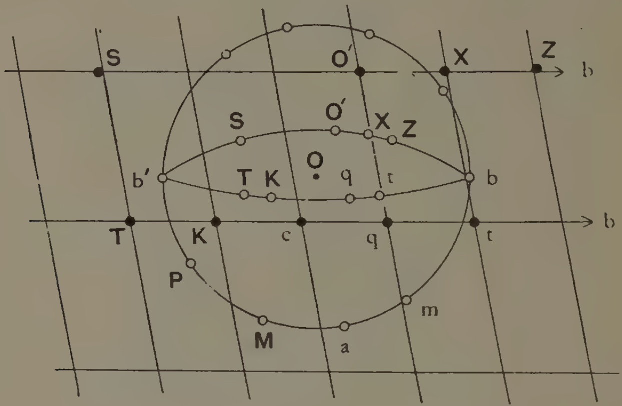


FIG. 55.

naturally includes the poles of a vertical pair of faces (actual or potential), although these are not visible, being at infinity. Thus, in Fig. 55, representing a crystal of copper sulphate, the zones $TKqt$ and $SO'XZ$ include the vertical face $b(010)$ at infinity; the zone cO' includes the vertical face $M(1\bar{1}0)$; the zone CZ, the vertical face $P(1\bar{3}0)$, and so on. Further the various poles (actual or potential) are strung along each zone-line at regular intervals. Now, *any three poles, such as* K, c, q, *which exhibit equal intervals* (Kc = cq), *together with the corresponding fourth pole at infinity, present the case of the harmonic quartette.* An inspection of the projection accordingly reveals the existence of the following harmonic quartettes (in which the indices of potential but undeveloped faces have been inserted without any significant letter): $btqc$, $bqcK$, $bcKT$, $btcT$, $bZXO'$, $bXO'(\bar{1}01)$, $bO'(\bar{1}01)(\bar{1}\bar{1}1)$, $b'(\bar{1}01)(\bar{1}\bar{1}1)S$, $bZO'(\bar{1}\bar{1}1)$, $bX(\bar{1}01)S$, $M(1\bar{1}1)cO'$, and so on.

Test 3.—The third test is purely numerical. Suppose we have to deal with four poles $A(010)$, $B(110)$, $C(100)$, $D(1\bar{1}0)$, and it is desirable to discover whether they are harmonic or no. Now, it may be found on inspection that *the indices of* B *term for term are the sum of the indices of* C *and* A, *and the indices of* D *are the difference.* Thus $B = C(100) + A(010) = (110)$; $D = C(100) - A(010) = (1\bar{1}0)$. Whenever this simple relation holds, *the four poles are harmonic.* Following is another important typical case: $A(110)$, $B(111)$, $C(001)$, $D(\bar{1}\bar{1}1)$.[16] The practical importance of acquiring the instinct for the immediate recognition of the harmonic case can scarcely be over-estimated, for with practice it is possible to evaluate an equation of the form $\cot x + \cot y = 2 \cot z$ in a minute's time (granted a five-figure table of natural cotangents is available).

The above discussion of the harmonic case is based essentially on instruction received in Petrograd. The principles have of course long been known, but the earliest paper (emphasising the importance of the harmonic case) known to the writer is one by E. S. Fedorov (*J. Min. Soc. Petrograd* [Russian] 1906, **44**, 201; abstracted in *Zeitsch. Kryst. Min.*, 1909, **46**, 202).

Note on the Anharmonic Case.—It is interesting (and sometimes useful) to note that the value p/q of formulæ (2) and (3) on p. 61 is discernible at a glance at the gnomonic projection (*granted as before that the pole* A *lies at infinity*). Thus, suppose the four poles A, B, C, D are the poles b, t, c, K of Fig. 55. In order to deduce the value p/q we have simply to measure the lengths (Kc, Kt) **from that terminal pole which is in sight** (in this case, K), and we find $p/q = Kc/Kt = 1/3$. Then putting these values into the general formula $p \cot AB - q \cot AC = (p - q) \cot AD$, we have $\cot bt - 3 \cot bc = - 2 \cot bK$. Similarly in the case of $bZXS$, $p/q = SX/SZ = 4/5$; whence, $4 \cot bZ - 5 \cot bX = - \cot bS$. Similar considerations hold for the vertical zone, but the 'stereographic diagram' of Fig. 54 has now to be invoked. In the case of the four poles $ABCE$, for example, the value p/q is seen to be $2/3$ (this being the ratio of the intercepts made by E, C and E, B on the line K); whence it follows that $2 \cot AB - 3 \cot AC = - \cot AE$.

5. **Application to Two-Circle Measurements.**—The Millerian formulae in their general and special aspects are of course applicable to two-circle readings resulting from a face-adjustment, for such readings are respectively the interfacial angles of classical goniometry and the interzonal angles of classical calculation. But it does not seem to be known that the formulae are equally applicable to the readings of four tautozonal faces resulting from a zone-adjustment.[17] It is, indeed, possible to compute a third ϕ-reading from two 'fundamental' ϕ-readings by the converse form of the Millerian formula, *i.e.* by the simplest,

[16] It is possible to generalise this test and, incidentally, make it practically useless. Thus, four poles A, B, C, D, are harmonic when $B = nC + mA$, and $D = nC - mA$. The special case discussed above is that in which $m = n = 1$. The following is really an example of the harmonic case: $A(120)$, $B(870)$, $C(210)$, $D(4, -1, 0)$. The example is quite fantastic, but if it actually occurred it would be a waste of time for most people to attempt to prove it to be harmonic, by the application of the generalised test. The obvious procedure is to evaluate the Millerian formula in the classical way and so finally deduce that $\cot AB + \cot AD = 2 \cot AC$. As a matter of fact, $m = 2$, $n = 3$.

[17] Although this relation was 'discovered' independently (and proved to be true by a consideration of spherical triangles) by the author in 1916, no claim for originality is made, since he subsequently became aware of the fact that its truth in principle is well known to mathematicians, and that a proof is actually included in Story-Maskelyne's *Crystallography*, p. 62 (1895). A diligent search in the crystallographic literature has failed to reveal any application of the principle to two-circle goniometry.

non-rectangular formula of crystallography. The ϕ-readings must of course be reckoned from the face pole in which the zone cuts the primitive; an arithmetical reduction has therefore to be effected when-even this pole is any other than $b(010)$.

Example.—Notwithstanding the liability to error mentioned on a previous page, suppose the ϕ-values of $q(011)$ and $K(0\bar{1}1)$—respectively $31^\circ 54'$ and $155^\circ 24'$—are given as fundamental angles in copper sulphate. Required the computed ϕ-values of $t(021)$, $c(001)$ and $T(0\bar{2}1)$. The five ϕ-values in question are represented in Fig. 56 by bw, by, bv, bx and bz respectively. In view of the harmonic relations we have :—

(1) $2 \cot bx = \cot bw + \cot by = \cot 31^\circ 54' + \cot 155^\circ 24' = \cot 31^\circ 54' - \cot 24^\circ 36' = 1{\cdot}60657 - 2{\cdot}18419 = -0{\cdot}57762$; whence $\cot bx = -0{\cdot}28881 = \cot 106^\circ 7'$.

(2) $\cot bv = 2 \cot bw - \cot bx = 2 \cot 31^\circ 54' - \cot 106^\circ 7' = 2 \cot 31^\circ 54' + \cot 73^\circ 53' = 2 \times 1{\cdot}60657$ (*i.e.* $3{\cdot}21314$) $+ 0{\cdot}28881 = 3{\cdot}50195 = \cot 15^\circ 56'$.

(3) $\cot bz = 2 \cot by - \cot bx = 2 \cot 155^\circ 24' - \cot 106^\circ 7' = -2 \cot 24^\circ 36' + \cot 73^\circ 53' = -2 \times 2{\cdot}18419$ (*i.e.* $-4{\cdot}36838$) $+ 0{\cdot}28895 = -4{\cdot}07943 = \cot (180^\circ - 13^\circ 46\frac{1}{2}') = \cot 166^\circ 13\frac{1}{2}'$.

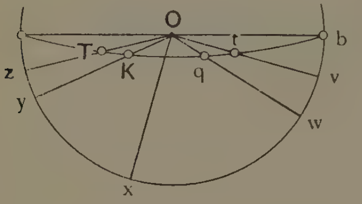


FIG. 56.

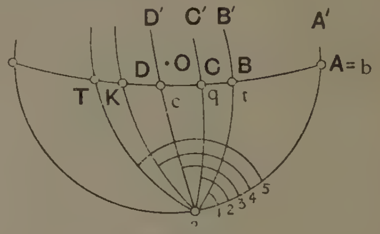


FIG. 57.

6. **Application to Coplanar ('Interzonal') Angles.**—Any application involving the substitution of four *zone-symbols* for facial indices requires no comment; but it is of considerable importance to note that there is no necessity to operate with zone-symbols, as the latter can be replaced by the indices of the four faces A, B, C. D, in which a fifth zone intercepts the zone-sheaf A', B', C', D' (see Fig. 57). [It should also not be lost sight of, that a glance at the indices A, B, C, D is sufficient to show whether the zone-circles A', B', C', D' are harmonic.] This well-known relation can be proved by establishing the identity of the two expressions

$$\left|\frac{\frac{A}{B}}{\frac{A}{C}}\right| \cdot \left|\frac{\frac{D}{C}}{\frac{D}{B}}\right| \quad \text{and} \quad \left|\frac{\frac{A'}{B'}}{\frac{A'}{C'}}\right| \cdot \left|\frac{\frac{D'}{C'}}{\frac{D'}{B'}}\right|$$

in general algebraic terms. The relation is of considerable interest to

single-circle workers. Thus suppose it were necessary to compute the various interzonal angles at the pole $a(100)$ of copper sulphate (in Fig. 57 these angles have been numbered 1, 2, 3, 4 and 5). Then a glance at the indices of the faces $b(010)$, $t(021)$, $q(011)$, $c(001)$, $K(0\bar{1}1)$, $T(0\bar{2}1)$, in which a zone bc intersects the given sheaf of zones, is sufficient to show that the following harmonic relations hold:

$$(1)\ \cot 1 + \cot 3 = 2 \cot 2;$$
$$(2)\ \cot 2 + \cot 4 = 2 \cot 3;$$
$$(3)\ \cot 3 + \cot 5 = 2 \cot 4;$$
$$(4)\ \cot 1 + \cot 5 = 2 \cot 3;$$

and any interzonal angle can therefore be readily computed without any preliminary establishing of zone-symbols by the somewhat tedious process of cross-multiplication.

The relation has a special field of application in Fedorov's method of 'bi-polar calculation.' A crystal is successively adjusted about a pair of faces A and B, the only angles measured being (1) the interzonal angles based on A and B, and (2) the interfacial angles in the zone AB. With the exception of one angle, the whole of the calculations consist of the evaluation of the cotangent formula relating to zone-circles (the interfacial angles having been discarded as being unnecessary). The method adopted by Fedorov (*J. Min. Soc. Petrograd* [Russian], 1906, **44**, 201; *J. School of Mines, Petrograd* [Russian], 1913, **4**, 325; *Zeitsch. Kryst. Min.*, 1914, **54**, 17) was to use zone-symbols in a somewhat circuitous manner. The classical method of replacing them by face-symbols is more direct. The method of measurement is interesting in two respects: (1) the angles measured are nearly wholly interzonal, and (2) the process involves less calculations than any other system except that described in Chap. VIII. The method is not applicable to single-circle goniometry. Measurement on the two-circle goniometer involves two separate adjustments. The three-circle goniometer permits the two sets of measurement to be made without any re-adjustment. It may be added that the principle of the method was independently deduced by Professor H. Hilton, who described it to the Mineralogical Society in the year 1921.

II. THE MULTIPLE TANGENT TABLE.

If the terminal poles (or zone-axes) A and D of the sequence A, B, C, D are rectangular, then the cotangent formula evidently undergoes a substantial simplification, for, as $\cot AD = \cot 90^\circ =$ zero, we get:

$$\frac{\cot AC - \cot AD}{\cot AB - \cot AD} = \frac{\cot AC}{\cot AB} = \begin{vmatrix} \overline{A} \\ \overline{B} \\ \overline{A} \\ \overline{C} \end{vmatrix} \cdot \begin{vmatrix} \overline{D} \\ \overline{C} \\ \overline{D} \\ \overline{B} \end{vmatrix} = p/q; \quad \ldots\ldots(1)$$

or, in the reciprocal form of tangents:

$$\frac{\tan AB}{\tan AC} = p/q;$$

or, as this is generally written:

$$\tan AB = (p/q) \tan AC; \quad \ldots\ldots(2)$$

The 'multiple tangent table' (p. 137 *sqq.*) gives the solution of this

equation for all the values of AB, AC, p and q that are likely to occur frequently in crystallography. It is a species of 'ready reckoner.'[18]

1. **Its Arrangement.**—The following excerpt will facilitate the description of the table:

1	2	3	4	5
15°00′	28°11′	38°48′	46°59′	53°16′
15°03′	28°16′	38°53′	47°05′	53°21′
15°06′	28°21′	38°59′	47°11′	53°27′

To take the first horizontal line. We see that $2 \tan 15° 0' = \tan 28° 11'$, $3 \tan 15° 0' = \tan 38° 48'$ and so on; $3/2 \tan 28° 11' = \tan 38° 48'$, $5/4 \tan 46° 59' = \tan 53° 16'$ and so on; $1/2 \tan 28° 11' = \tan 15° 0'$; $2/5 \tan 53° 16' = \tan 28° 11'$ and so on. Although only four calculations were originally involved, the line really summarises the results of twenty possible operations. The same is true for all other horizontal lines in the table.

It will be observed that entries in the first column differ regularly by 3′. Experience shows that this is quite sufficient, for intermediate values can be readily deduced by inspection.[19] Thus the differences in this region of the table are 5′ in the second column, 5′ or 6′ in the third, fourth and fifth columns. Results taken by difference are therefore generally true to the nearest minute; in individual cases the error may be 1′. Now mean measurements from different crystals differ from each other considerably (*cf.* G. Wulff, *Zeitsch. Kryst. Min.*, 1904, **38**, 1), and the mean values adopted as a result of measurement of several crystals by independent, careful workers are by no means identical, differing as they do within the range indicated by 6′ or even 10′.

[18] During the proof-correction my attention has been drawn to a paper contributed by A. Hutchinson to the *Napier Tercentenary Memorial Volume* (Edinburgh, 1915, p 329) containing some new devices, one of which permits of a rapid solution of the equation $\tan x = p/q \tan y$, with a degree of accuracy that is only surpassed by formal methods of calculation. The apparatus consists of two strips of cardboard (about 20 cms. long), each carrying a printed scale along its edge: in one case, a logarithmic scale of tangents, and in the other case a logarithmic scale of various, simple p/q— values. Its method of application can be described in terms of the prism zone of the orthorhombic mineral anglesite, in which $am = 38°8'$. The two scales are first mutually adjusted, by sliding one edge along the other, until the value 38°8′ (approx.) of the tangent scale is opposite the mark 1 of the p/q— scale. Corresponding tangents and p/q— values are then in conjunction for every other possible face between a (100) and b (010), and the indices of any face may therefore be read from a measured angle, or conversely the angular value from the indices. Practical trials of this simple form of slide-rule (kindly presented by Dr. Hutchinson) gave results of a higher degree of accuracy than can be easily obtained by any graphical method, the maximum error being 9′. The degree of accuracy is naturally increased if a large scale instrument be employed, the maximum error recorded by Dr. Hutchinson himself for a 50-inch instrument being 4′—a value which happens to coincide with the average error as obtained by the present writer with the small scale instrument.

[19] It is, perhaps, an open question whether the earlier part of the table should not be given minute by minute in view of the considerable differences involved in the 3rd, 4th and 5th columns. The arguments in favour of the present form are: (1) uniformity; (2) the belief that the 4th and 5th columns can never have any extended applicability; and (3) the belief that the same is true of the earlier parts of the table *as a whole*, that is of those parts where the differences are greatest.

The table would therefore seem to meet all reasonable requirements in the province of crystal measurement.

Although only such fractional multipliers as involve the numbers 1—5 are directly represented, the table has of course further possibilities. Multiplication by six, for example, can be effected in two stages (3×2, or 2×3), the observer moving to a different part of the table; a somewhat fantastic multiplication by 9/4 could be effected by $3 \times 3/4$, and so on. Multiplications involving prime numbers higher than five are not provided for, since they occur so rarely in crystallography. A few remarks on this point may not be out of place.

The addition of further columns giving the results of a multiplication by six, seven, eight and nine, or alternatively, the computation of an entirely new nine-column table based on the primes lying between 1 and 19 inclusively, would make it possible to effect 72 direct operations with each horizontal line (and almost any fractional multiplication whatsoever by successive consultations of the table). Such a generalised multiple tangent table would no doubt be of *occasional* value in the special province of mineralogy, but so far as the general requirements of crystallography are concerned, cases involving numerical co-efficients 4 and 5 are already of very rare occurrence. Accordingly the fourth and fifth columns have been dispensed with altogether in the last pages of the table, especially as these are but thinly represented by crystals. They were, as a matter of fact, added as an after-thought in order to provide for results which must inevitably follow any eventual improvement of technique, permitting the measurement of platy or scaly crystals—a province in which angles close to 90° are to be expected on structural grounds (any correspondingly successful extension to needles are sufficiently covered by a table commencing at 5°).

The table was first computed for differences of ½° (reproduced as the 'supplementary table' of p. 149). As the labour of computation was much less than was anticipated, the table was then expanded into its present form. The various values were computed from the first column (and not from the third), because the most frequent cases in practical work involve whole number multiplications or divisions. As the table from its nature must necessarily sometimes give a value which is 1′ removed from the 'true' value, it was thought desirable to place this 'burden of error,' however trivial, on such rare operations as a multiplication by 2/5, 5/2, 4/5 or 5/4, and not on the most frequent operations of all, namely, multiplications by 1/2 or 2.

2. **Its Dual Uses.**—The table can be used in two ways, one the converse of the other, as implied by the two equations:

$$\tan AB = (p/q) \tan AC \quad \text{.......................(1)}$$

$$p/q = \tan AB/\tan AC \quad \text{.......................(2)}$$

A little reflection will show that equation (1) refers to precise calcula-

tions of the theoretical value of an angle, and that equation (2) implies the substitution of the table for a graphical method in the determination of indices. It seems expedient to consider the two applications of the table in the above-mentioned order, if only to emphasise its more important function, namely, to obviate precise calculations almost wholly in the rectangular systems.

3. **Its Employment in Formal Calculations.**—When used in this way the table evidently presupposes a knowledge of the numerical coefficient p/q. It will be shown later how the value of p/q can be read off the gnomonic projection; but as a reader may not be fully conversant with this projection, it will be presupposed that he has determined the value by the formal evaluation of the Millerian formula or by the simple slide-rule method introduced by Hutchinson. In the following example (taken from the single-circle section of the author's forthcoming monograph on goniometry) the values of p/q are always the simple numbers 2 or 3, and can almost be assumed to be discernible 'intuitively.' It may be added that these simple numbers 2, 3 (rarely 4) and also 2/3, are almost the only multipliers or divisors that occur in practice.

If topaz were a new mineral or laboratory product, it would be desirable to investigate it crystallographically and place its constants on record. An actual crystal is shown in Fig. 58, which, having been

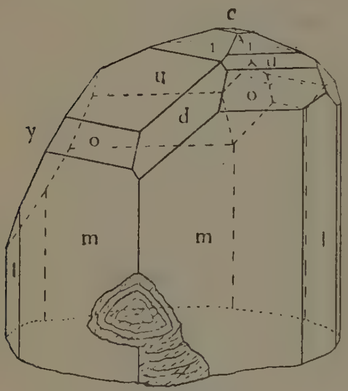


FIG. 58.

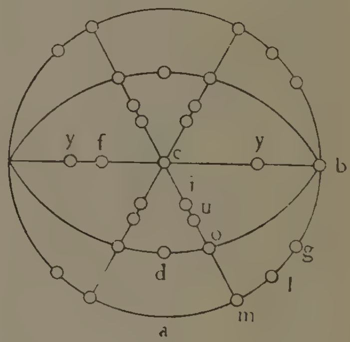


FIG. 59.

drawn from a plan by means of the gnomonic projection, is true to nature, except that two or three faces are too small for representation. It should be noted that the single-circle measurements, implied by the zones drawn in Fig. 59 are sufficient to establish both the geometrical relations of the faces and the system. With regard to the indices, we have a right to take o as the parametral plane (111), for, as the mineral is assumed to be new, it is open to us to make that selection which leads to the

simplest set of indices. It is now presupposed that the various indices have been determined to be: *m*(110), *l*(120), *g*(130), *b*(010), *c*(001), *f*(011), *y*(021), *d*(101), *i*(113), *u*(112), *o*(111). In view of the imperfect reflections from the prism faces, we will take as fundamentals the mean measured angles $co = 64° 2'$ and $do = 24° 53'$. The calculation of the axial ratios is the next step. By solving the right-angled triangle *cdo*, we find $dco = am = 27° 54'$; its tangent yields $a : b = 0{\cdot}5296$. The solution of the right-angled triangle *cof* gives the value $cf = 43° 52'$; its tangent yields $c : b = 0{\cdot}9611$ (these values may be checked by the tangent-cotangent scale of the crystallographic protractor).

The multiple tangent table now comes into use. To derive the theoretical value of *cy* we have to evaluate the equation $\tan cy = 2 \tan cf = 2 \tan 43° 52'$. We consult the table and find: $43° 51'$ (first column) $\rightarrow$ $62° 30'$ (second column); also, $43° 54' \rightarrow 62° 33'$. It is therefore clear that $43° 52' \rightarrow 62° 31'$. This is the angle required. We will now consider the vertical zone and incidentally assume that any requisite differences in the multiple tangent table have been allowed for. Setting out from the value $am = 27° 54'$, we find: $27° 54'$ (first column) $\rightarrow$ $46° 38'$ (second column) $\rightarrow$ $57° 48'$ (third column). These are the computed values of *al* (100 : 120) and *ag* (100 : 130) respectively. Now consider the zone *co*. The indices of *i* and *u* have been previously determined to be (113) and (112) respectively. The angles to be computed are smaller than the initial angle $co = 64° 2'$. For the angle *ci* (001 : 113) we must therefore consult the third column at the value $64° 2'$ and work back through the second to the first columns; thus, $34° 23' \leftarrow [53° 51' \leftarrow\ ]\ 64° 2'$. For the angle *cu* (001 : 112) we consult the second column at the same value $64° 2'$, and pass back to the first column; thus, $45° 45' \leftarrow 64° 2'$.

This completes the calculations. Granted a little experience with the new implement, a researcher should not need half-an-hour for the whole of the calculation—always provided he has refrained from measuring superfluous zones.

4. **Derivation of the Co-efficient 'p/q.'**—The fundamental Millerian formula must, perforce, always be invoked in rare cases of great difficulty; but in 99 cases out of a hundred substances (or, say, 999 cases out of a thousand angles) a glance at the gnomonic projection is sufficient (the rare, exceptional case being of course that in which, owing to highly complex indices the projection loses its definitive value). Consider the gnomonic projection of topaz (Fig. 60). The initial use of the projection is of course to determine the indices, but we are here concerned with the stage of computations. In the gnomonic projection of a rectangular system like the orthorhombic, the distance of any face-pole from the centre is a measure of the tangent of the angle between

that face and the basal plane. The ratio of a pair of such tangents is therefore the ratio of the pair of central distances, and is the ratio p/q that is wanted. Thus it can be seen immediately from the diagram that

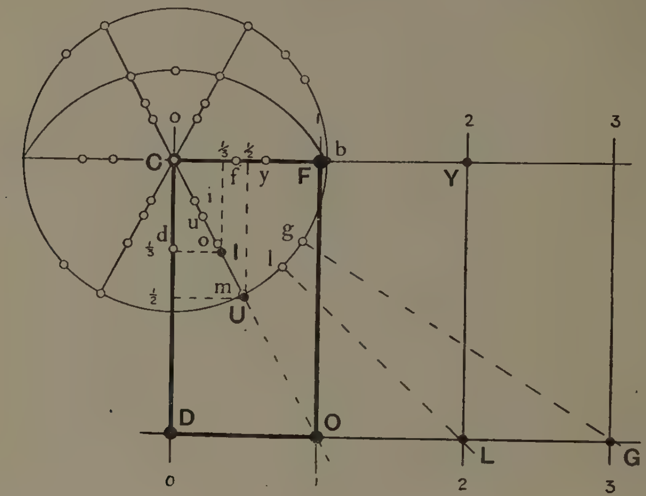


FIG. 60.

tan cy = 2 tan cf, because the gnomonic pole of Y is *twice* as distant from the centre as the pole F; and that tan $cu = \frac{1}{2}$ tan co, because the distance of the pole U is half that of the pole O, and so on. It must

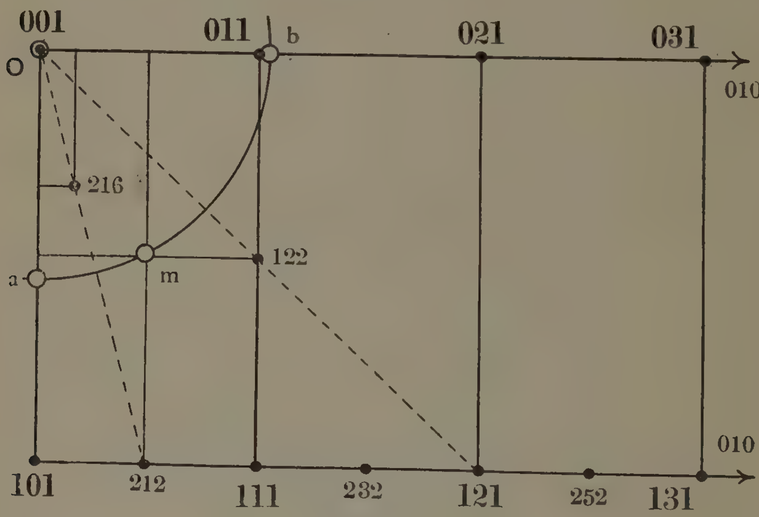


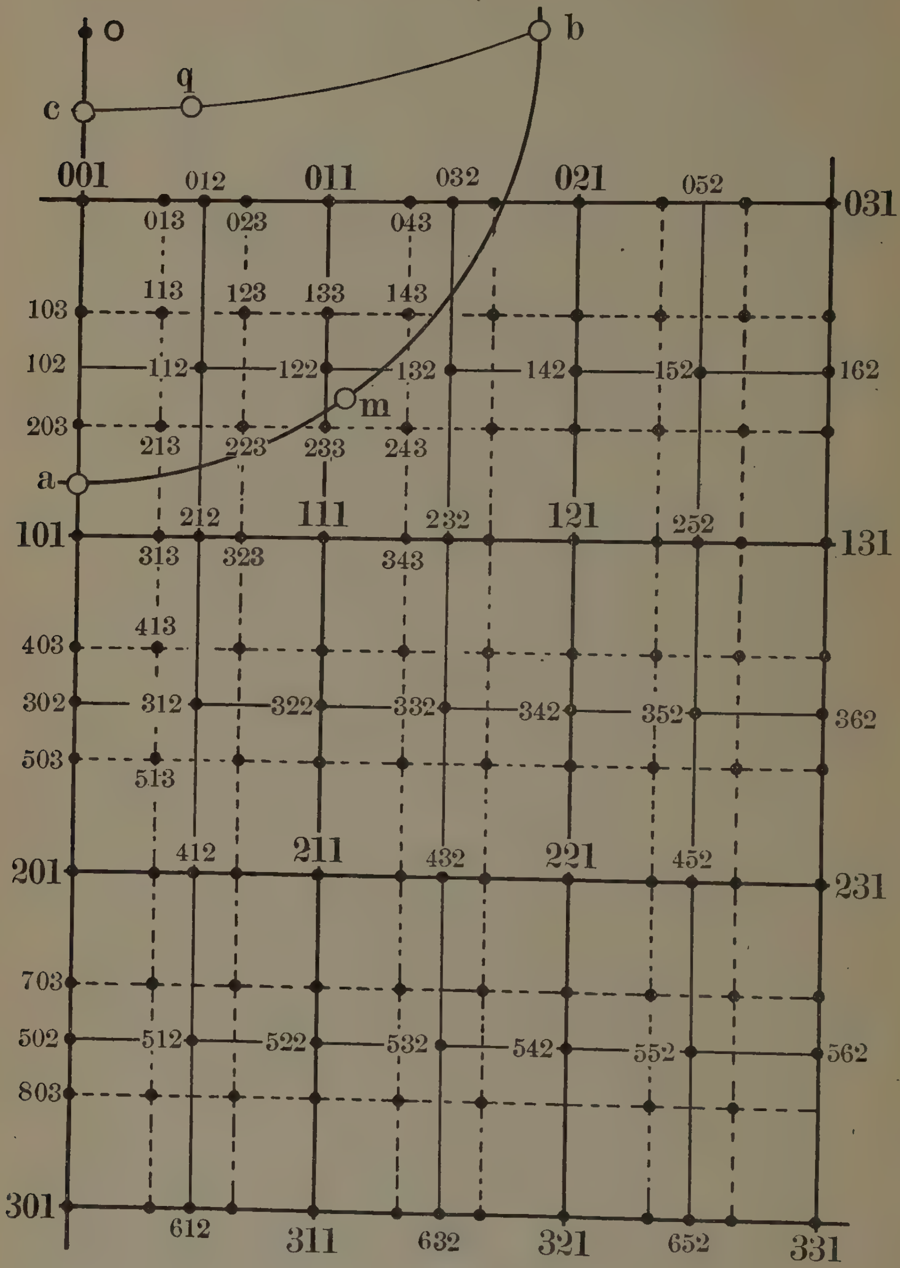
FIG. 61.

be noted that the gnomonic projection provides the key even in a case in which the angles actually measured pass through $b(010)$ or $a(100)$, as opposed to $c(001)$. The regular steps along the gnomonic zone-line,

(101)—(212)—(111)—(232)—(121)—(252)—(131) → (010) of Fig 61, for example, are enough to indicate that tan (101 : 212) = $\frac{1}{2}$-tan (101 : 111); that tan (101 : 232) = 3/2 tan (101 : 111); that tan 101 : 121) = 2 tan (101 : 111); and so on—*the angles all being reckoned from that face pole which is perpendicular to the pole at infinity* (a necessary condition). Similarly, tan (011 : 122) is deduced to be equal to $\frac{1}{2}$ tan (011 : 111) from the two circumstances (1) that the pole (011), from which the angles are reckoned, is perpendicular to the pole at infinity [in this case *a*(100)], and (2) that the linear distance (011)—(122) is half the distance (011)—(111). Finally, to revert to a zone passing through *c*(001), we have tan (001 : 122) = $\frac{1}{2}$ tan (001 : 121); and so on. It will thus be seen that the value of the numerical coefficient p/q can be read directly from the gnomonic projection. Moreover, the projection informs us whether we are seeking for a larger or a smaller angle than the fundamental; whether, for example, we have to make for column 3 and move to column 2, or alternatively make for column 2 and move to column 3. It is now, perhaps, advisable to recapitulate in terms of an extreme case. If our measured crystal of topaz had happened to develop faces (216) and (212), we should have already plotted their poles on a gnomonic diagram at an earlier stage of the work, in order to determine the indices. From this projection we see immediately that tan (001 : 216) = 1/3 tan (001 : 212). Consequently, if it be presupposed that the angle 001 : 212 had been computed from a right-angled triangle, the value 001 : 216 would follow immediately from the multiple tangent table, *i.e.* without any necessary recourse to cross-multiplication of indices followed by a logarithmic evaluation. The values quoted in Goldschmidt's *Winkeltabellen* are 61° 49½′ and 31° 53½′. That they are mutually consistent (even to the half minute) is obvious from a glance at the multiple tangent table.

As regards the vertical faces: it is well known that tan (100) : (*hk*0) = (*k*/*h*) tan (100 : 110) (*notice the inversion*). Alternatively, the value p/q may be read from the gnomonic projection by replacing mentally each vertical face (*hk*0) by its 'directrix' (*hk*1); or again, by carrying out the stereographical construction (p. 42). One of these operations will of course have already been carried out in the preliminary determination of indices.

5. **The Monoclinic System.**—This system is statistically more important than any other, and it is therefore all the more important to note that the gnomonic projection preserves its usefulness as a guide to the application of the multiple tangent table, provided it be always remembered that rectangular zones are now restricted to those that pass through *b*(010). A typical monoclinic projection is that reproduced as Fig. 62: actually on the scale of colemanite, but not a faithful representation of the forms described by A. S. Eakle (*Bull. Dept. Geol.*,


o
b
q
c
001
012
011
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703
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552
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803
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612
311
632
321
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331

Fig. 62.

Univ. of California, 1902) in a lucid presentation of two-circle goniometry (as advocated by V. Goldschmidt), which has fortunately become more readily accessible as a reprint in T. W. Walker's *Crystallography* (1914). A glance at Fig. 62 is sufficient to show that tan (001 : 021) = 2 tan (001 : 011). These interfacial angles of single-circle goniometry are the 'eta-angles' of Eakle's description, and a glance at the multiple tangent table will, indeed, show that tan 45° 34′ (or, better, 45° 33½′) is equal to 2 tan 27° 1′. Similarly, on the zone line (101)—(111) → we have a series of forms developed on colemanite, namely, (101)—(111)—(232)—(121)—(131)—(141), and the regular intervals indicate that tan (101 : 232 = 28° 36′) = 3/2 tan *ro* (101 : 111 = 19° 58′); tan (101 : 121) = 2 tan *ro*, and so on. It is also important to note that the horizontal lines of the front portion of a monoclinic gnomonic diagram as presented by Fig. 62, are a sure guide to corresponding zonal relations at the back of the crystal (although the actual angles are different). Thus, the front line (301)—(311)—(321)—(331) of the diagram, indicates equally well that tan ($\bar{3}$01 : $\bar{3}$21) = 2 tan ($\bar{3}$01 : $\bar{3}$11), and that tan ($\bar{3}$01 : $\bar{3}$31) = 3 tan ($\bar{3}$01 : $\bar{3}$11); and the multiple tangent table at the entry 14° 23′ (first column) → 27° 9′ → 37° 34′ reproduces Eakle's values. Similarly, the well-defined horizontal zone-line (201)—(211)—(221)—(231) is a sufficient reminder that the same 1, 2, 3 equal steps will be true of the line ($\bar{2}$01)—($\bar{2}$11)—($\bar{2}$21)—($\bar{2}$31), and therefore that Eakle's values for the three angles as measured from ($\bar{2}$01) are mutually deducible by the help of the multiple tangent table (specifically, 19° 52′ → 35° 51′ → 47° 18½′). That the angles of the vertical zone of a monoclinic crystal are also mutually deducible needs no special emphasis, since this zone, passing through *b*(010), is also rectangular. For example, the angles which (110), (210), (310) in colemanite, respectively make with *b*(010) are given by the entry: 53° 53½′ → 69° 57½′ → 76° 20′.

This exhausts a single-circle worker's practical interest in the multiple tangent table, but the two-circle worker can make further applications. In the monoclinic system Goldschmidt's 'eta-subscript-zero' angles (η_0) are of course true interzonal angles [the angle η_0 for (011), for example, represents the angle, whose tangent is equal to the axial ratio *c* : *b*]; and it follows that tan η_0 (021) = 2 tan η_0 (011). Specifically in colemanite, tan 47° 22′ = 2 tan 28° 30′. But it is in the province of the so-called 'non-crystallographic' phi-readings of zones passing through *b*(010) that the table acquires a really practical, two-circle importance. Here, again, the gnomonic projection is an infallible guide to the correct *p*/*q* co-efficient. It is convenient to reckon the phi-values complementarily, *i.e.* from *a*(100) or from the other proper pole lying in the zone *ac* instead of from *b*(010)—because the latter pole

lies at infinity—and finally effect a reduction. To take once more the poles in colemanite spaced along the zone-line (101)—(111) →. Eakle's standard phi-value for (111) is $63° 56\frac{1}{2}'$. Its complementary value, then, is $26° 3\frac{1}{2}'$; and the gnomonic diagram directly indicates that :

(1) $\tan [90° - \text{phi}(232)] = 3/2 \tan 26° 3\frac{1}{2}' = \tan 36° 15'$; *i.e.* $\text{phi}(232) = 53° 45'$.

(2) $\tan [90° - \text{phi}(121)] = 2 \tan 26° 3\frac{1}{2}' = \tan 44° 22'$; *i.e.* $\text{phi}(121) = 45° 38'$.

(3) $\tan [90° - \text{phi}(131)] = 3 \tan 26° 3\frac{1}{2}' = \tan 55° 43\frac{1}{2}'$; *i.e.* $\text{phi}(131) = 34° 16\frac{1}{2}'$.

(4) $\tan [90° - \text{phi}(141)] = 4 \tan 26° 3\frac{1}{2}' = \tan 62° 55\frac{1}{2}'$; *i.e.* $\text{phi}(141) = 27° 4\frac{1}{2}'$.

Eakle's values are $53° 44\frac{1}{2}'$, $45° 38'$, $34° 17'$ and $27° 5'$.

It is, perhaps, worth while repeating that the phi-values of zones, other than those passing through *b* (010), obey the *general* Millerian cotangent formula, no matter to what system the crystal belongs. A particular case is the monoclinic zone *mc* (110 : 001). In any case in which the zone is richly developed it is advantageous to develop the phi-readings (as reckoned from *m*) by the cotangent formula, the multiple tangent table being no longer applicable.

6. **The Hexagonal System.**[20]—The only difficulty (and that a small one) which necessarily accompanies the application of a two-co-ordinate gnomonic diagram to a 4-index description of a crystal is restricted to the preliminary construction of the primitive gnomonogram. When once this has been effected, the determination of indices is as simple as it is in the orthorhombic system. With regard to the special form of symbolisation use will be made of Bravais' indices, as they are almost

[20] This and the following section on the rhombohedral system might well be omitted on a first reading, as they are not merely concerned with the application of the multiple tangent table, but also with the general application of the gnomonic projection to these two systems. As indicated on a previous page the author's view is that no attempt should be made to master the principles of the gnomonic projection (and therefore of the multiple tangent table) so far as these two systems are concerned until all the other systems have been thoroughly studied. This should not be taken to mean that the two systems really present formidable difficulties; it merely implies that the adoption of four axes in the one case, and an unusual orientation of three axes in the other, make it necessary to proceed with a little circumspection. For the purely descriptive purposes of mineralogy, the two systems are frequently regarded from a unitary standpoint, the 'rhombohedral' 3-index system being, for example, exclusively used by Miller, and the 'hexagonal' 4-index system by Dana, Tschermak and Goldschmidt (in the latter case with not infrequent changes of orientation). From the purely descriptive point of view there is much to be said in favour of a unitary symbolisation (especially if four indices be adopted) both on general grounds and also because there are numerous cases in which it is not definitely known whether a substance is hexagonal or rhombohedral (it cannot of course be both). But from the structural standpoint both systems of symbolisation are necessary and in the great majority of individual cases one system is as elegant in application as the other is clumsy. The two systems will therefore be considered separately here, but without any attempt to discuss the criteria to be adopted in the proper application of either.

universally adopted.[21] The characteristic feature of these indices is that the sum of the first three is always equal to zero. It is therefore particularly easy to write in, by inspection, any third index which has been initially discarded. *For gnomonic purposes it is especially convenient to discard, and eventually restore, the* third *index*, as opposed to the first or second; so this procedure will be adopted.

1. Determination of Indices by the Gnomonic Projection.—A combined gnomono-stereogram of a hexagonal crystal (actually beryl) is shown in Fig 63, the forms developed being $c(0001)$, $p(10\bar{1}1)$, $m(10\bar{1}0)$ and $s(11\bar{2}1)$. The gnomonic poles are derived from the stereographic

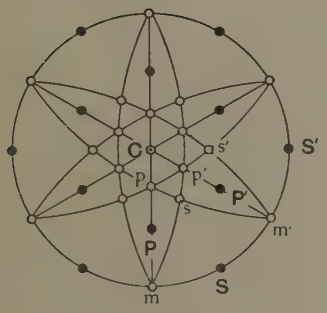

Fig. 63.

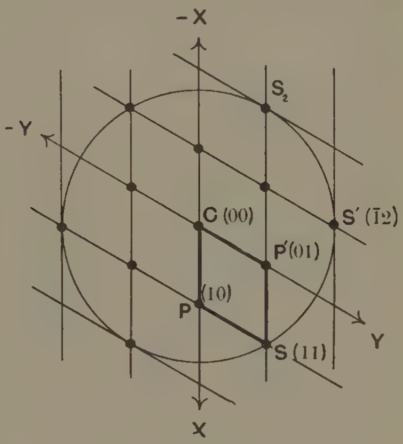

Fig. 64.

in the usual way by marking off twice the central angular distance. Since in beryl the angle *cs* happens to be close to 45° (in reality 44° 56′) the gnomonic pole necessarily lies on the stereographic primitive, but this has no general significance. As in all the previous cases the gnomonic projection is best applied to crystallographic purposes in terms of a pair of co-ordinate axes and corresponding unit-lengths. The four indices have therefore to be reduced to a pair of plane coordinates. This is effected in two stages: (1) the third, geometrically superfluous, crystallographic axis and index are discarded—thus, $c(00[0]1) \rightarrow c(001)$; (2) as in all the previous systems, the first and second indices are divided by the third—thus $c(001) \rightarrow c(0/1,0/1 \rightarrow c(00)$. When so treated, the four faces $c(0001)$, $p(10\bar{1}1)$, $s(11\bar{2}1)$ and

[21] The implication is that the variant adopted by Fedorov, in numerous papers and also in his unpublished 'Crystal Kingdom,' will not be adopted. The differences are: (1) That Bravais takes the positive directions of the first three crystallographic axes as being separated by angles of 120° in a counter-clockwise direction, whilst Fedorov prefers angles of 60° in a clockwise direction; and (2) that Bravais places the vertical axis (and its corresponding index) last, Fedorov first. In Fedorov's system of symbolisation, then, the first index refers to the unique axis, and the fourth index is equal to the second minus the third.

p' (01$\bar{1}$1) yield the primitive gnomonogram (00)—(10)—(11)—(01); and these four poles naturally determine both the directions and also the unit lengths of the two co-ordinates XOX', YOY' (shown in Fig. 64). These co-ordinate axes, which must not be confused with the original crystallographic axes, permit a straightforward determination of the plane co-ordinates of any other gnomonic pole, the corresponding indices being derivable therefrom by the simple method of the following table (the derivation of the indices of the pole S_2 is left to the reader as an *Exercise*):—

Gnom. Pole (Fig. 64) and plane co-ord.	Add 1 as final index.	Clear up fractions if necessary.	Interpolate 3rd index.	Indices.
P (10)	(101)	(101)	(10[$\bar{1}$]1)	(10$\bar{1}$1)
P'(01)	(011)	(011)	(01[$\bar{1}$]1)	(01$\bar{1}$1)
S ($\bar{1}$1)	(111)	(111)	(11[$\bar{2}$]1)	(11$\bar{2}$1)
S'($\bar{1}$2)	($\bar{1}$21)	($\bar{1}$21)	($\bar{1}$2[$\bar{1}$]1)	($\bar{1}$2$\bar{1}$1)
$(h/k, l/m)$	$(h/k, l/m, 1)$	(hm, lk, km)	$(hm, lk, [-hm-lk]km)$	

It need scarcely be added that the converse problem is equally easy. If any pole with indices $(pqrs)$ be given, its gnomonic pole can be inserted into the diagram as follows. The third index r is discarded; p and q are each divided by s, and the quotients p/s, q/s represent the plane co-ordinates of the pole in terms of the axes, XOX' and YOY' of Fig. 64, and their unit lengths, $CP = CP'$. When once this gnomonic pole has been inserted the stereographic pole follows immediately by marking off a point at half the central angular distance.

With regard to the vertical zone it need only be stated that the indices can be determined in the two usual ways: (1) *gnomonically*, the indices of a pole $(pq[r]0)$ being defined by those of its directrix $(pq[r]1)$; or, (2) *stereographically*, by discarding the third index throughout, drawing a convenient line parallel to the diameter carrying the stereographic pole '(010),' *i.e.* (01[$\bar{1}$]0), and comparing the intercepts with the unit intercept '0 — 1,' as explained on p. 43 in terms of Figs. 45-46.

2. COMPUTATION OF ZONAL ANGLES BY THE MULTIPLE TANGENT TANGENT TABLE.—As indicated previously, under the orthorhombic and monoclinic systems, a gnomonic projection is a complete guide to the proper numerical co-efficient p/q, provided it is thoroughly grasped which zones are rectangular. Thus from any general hexagonal gnomonic diagram, corresponding to that of Fig. 62 which holds for the monoclinic system (such a general hexagonal diagram is not reproduced here), it is obvious at a glance that tan (0001 : 20$\bar{2}$1) = 2 tan (0001 : 10$\bar{1}$1)); that tan (0001 : 11$\bar{2}$2) = $\frac{1}{2}$ tan (0001 : 11$\bar{2}$1); and (to take an apparently but not really more complicated example) that tan (10$\bar{1}$1 : 22$\bar{4}$3) = 1/3 tan (10$\bar{1}$1 : 02$\bar{2}$1). It should be noted that the

multiple tangent table is *not* readily applicable to the vertical zone as the faces ($10\bar{1}0$) and ($01\bar{1}0$), *i.e.* the faces '(100)' and '(010)' are inclined at 60°, instead of 90°, in this system. These angles are, however, independent of the substance, and their values for every form that is likely to occur are given in Dana's *System*, p. xxx. Alternatively, they can be readily computed by the *general* Millerian formula. So far as the investigator of laboratory products is concerned, the value ($10\bar{1}0$) : ($21\bar{3}0$) = 19° 6′ may occasionally be of use.

7. **The Rhombohedral System.**—The gnomonic treatment of this system is fraught with difficulties of a higher order than in the case of the hexagonal system—owing to the fact that no crystallographic axis is any longer perpendicular to the plane of the diagram. Now there are two ways in which a crystallographic axis could be brought into the perpendicular. The first method would be to rotate the stereographic projection about a zone *rr* (100 : 010) until it comes into the vertical, whereby it would acquire the aspect of a monoclinic crystal. The second method would consist of a recourse to a hexagonal symbolisation for the actual determination of indices, and an eventual transformation of ($pqrs$) into (hkl) by means of the equations :

$$h = p + 2q + s; \quad k = -2p - q + s; \quad l = p - q + s.$$

But neither of the above methods can be recommended for the following reason. There is no point in applying the gnomonic projection to the determination of indices unless there are numerous forms to determine; and if there are numerous forms, the transformation *either* of poles *or* of their symbols becomes a relatively serious business. It therefore seems better to meet the difficulties rather than evade them; to treat the crystal rhombohedrally and not as a monoclinic or hexagonal counterfeit. This was the attitude adopted by Mallard, the pupil of Bravais, in his profound studies of crystal-structure.

Although the essentials of the following gnomonic treatment are taken from p. 125 of Mallard's *Traité*, the orientation of Lewis and Miers will be adopted, as having a more systematic relation to the generally accepted, hexagonal notation[22] (each rhombohedral axis lying in a vertical plane, normal to that which contains the corresponding axis of the hexagonal notation).

[22] Lewis *Crystallography*, 1899; Miers' *Mineralogy*, 1902 [a revised edition by H. L. Bowman in preparation]—translated into French by O. Chemin: *Manuel pratique de Minéralogie*, 1906. In this orientation (also adopted by Viola in his *Trattato di Cristallografia*, 1920) the phi-values of (100), (010) and (001) are respectively 30°, 270° and 150°; whilst in that of Miller (followed by Mallard, de Lapparent, Friedel and Story-Maskelyne) the values are 30°, 150° and 270°; in that of Groth (followed by Liebisch and Tutton), 90°, 330° and 210°; in that of Fedorov, 330°, 210° and 90°; and in his later work (*e.g. The Abridged Course of Crystallography* [Russian], 1910, the new values 150°, 30° and 270°. Tschermak, and Moses and Parsons employ a hexagonal notation throughout.

Considerable difficulties were encountered in finding a suitable example for illustrative purposes. Numerous rhombohedral laboratory products had to be ruled out either on account of a too-simple form-development or because the angle *cr* (111 : 100) was too great for a clear reproduction of the gnomonic diagram. It will be realised that a cubic crystal, set up rhombohedrally, requires an unusual amount of space, since the angular value *oa* (111 : 100) = 54°44′ now plays the dominant rôle (instead of the angle *ad* (001 : 101) = 45°. With most rhombohedral substances a primitive circle of 3 cms. radius would be preferable to one of 5 cms. for gnomonic purposes. Fortunately the angle *cr* (111 : 100) = 27°20′ is very small in tourmaline, which circumstance makes it more suitable than calcite or even phenacite as an illustration.

1. DETERMINATION OF INDICES BY THE GNOMONIC PROJECTION.—The stereographic projection of a rhombohedral crystal (tourmaline) is given in Fig. 65, the forms developed being *r*(100), *e*(101), *o*($1\bar{1}1$), *a*($1\bar{1}0$),

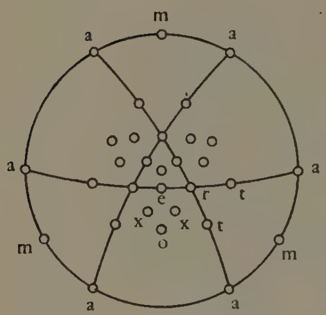

FIG. 65.

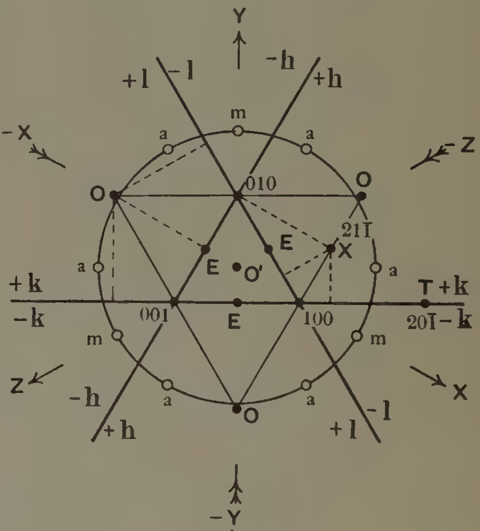

FIG. 66.

m($2\bar{1}\bar{1}$), *x*($2\bar{1}1$) and *t*($20\bar{1}$). The corresponding gnomonic projection, in which capital letters are used for gnomonic poles, is shown in Fig. 66. It is seen that the projection is not distorted—*i.e.* the poles do not map out trapezia but are arranged parallelogrammatically, or better stated according to a system of equilateral triangles. But the method of determining the indices of a gnomonic pole from its position in the diagram is not quite so simple as it was in the hexagonal system.

The equilateral triangle 100—010—001 must now be taken as the primitive gnomonogram. The three sides of this triangle are produced in both directions, and although not intersecting in a common point, are all three **taken as co-ordinate axes.** It should be noticed that these three co-ordinate axes are marked *plus* and *minus* *h*, *k* and *l*. The meaning of this is that if a gnomonic pole lies on the *plus* side of an axis, its corresponding index *h*, *k* or *l* is positive; and vice-versa. An examination of the diagram will show that a gnomonic pole which

happens to fall inside the primitive gnomonogram will have all three indices positive. [NOTE.—The letters X, Y, Z on the border of the Figure represent the crystallographic axes, in plan. They are really unnecessary in what follows].

The gnomonic projection is of course obtained from the stereographic by plotting off twice the central angular distance. *The indices of, say, the face* O ($\bar{1}11$) *are determined by comparing the lengths of normals drawn to the three co-ordinate axes from the pole in question* (*cf.* the interrupted lines of Fig. 66). In this case the three lengths are evidently equal. This determines the indices to be equal; but as the gnomonic pole is on the negative side of the h-axis, and on the positive sides of the k and l-axes, the indices are ($\bar{1}11$). Similarly with the pole X. The normal to the h-axis is seen to have twice the lengths of the other two normals, which fact together with the circumstance that the pole is on the negative side of the l-axis, determines the indices to be ($21\bar{1}$). Similarly the indices of T will be made out to be ($20\bar{1}$), the middle index being necessarily zero as the pole actually lies on the k-axis and is therefore at a zero-distance from it. Again we have E(101). If developed, the basal plane, O', would on the same principles have the indices (111); and so on.

The converse problem of inserting a gnomonic pole into the diagram from its indices (hkl) is not so simple as in the hexagonal system. Although of little practical importance, Mallard's method will be given here on grounds of comprehensiveness. The indices (with due regard to sign) are first summed; thus $h + k + l = n$. The gnomonic pole required is that point which lies at the perpendicular distances (from the lines h, k, l, cf. Fig. 66) given respectively by ah/n, ak/n, al/n, where a represents the altitude of the triangular gnomonogram (say, the distance between the gnomonic poles of 010 and 101). Thus in the case of a face (4, 2, -3), not shown in Fig. 66, $n = 3$; and the distances of the gnomonic point required (from the lines h, k, l) are $4/3 \times a$, $2/3 \times a$ and $-a$ respectively. Any two of these three data are sufficient to fix the pole, but all three indices (hkl) are none the less really involved for they were needed in the determination of n.

The graphical treatment of the vertical zone does not seem to have been outlined by Mallard, but the application of the 'stereographic method' is perfectly straightforward. It is presupposed that we know the indices of the two common hexagonal prisms a($10\bar{1}$) and m($2\bar{1}\bar{1}$). [That the latter happens to be a trigonal prism in tourmaline does not concern us here, but it is unfortunate that the other trigonal prism ($11\bar{2}$) has not been inserted in Fig. 66. This omission, at any rate so far as the face ($11\bar{2}$) is concerned, should be rectified by pencilling it in (at the 330°-phi position) as it is involved in what follows.] It will now be convenient for explanatory purposes to regard a second face, belonging to the form a($10\bar{1}$) as 'b($01\bar{1}$)'; so that we now have three faces, a($10\bar{1}$), m($11\bar{2}$) and b($01\bar{1}$)—occupying the 0°, 330° and 300° points on the primitive. Now in the vertical zone one of the three indices is superfluous, since the sum of the three is zero. We therefore discard the third index, and obtain a(10), m(11), b(01). The simple construc-

tion of Fig. 45, p. 43 is now applicable, the crystal being, as it were, an anorthic crystal in which *ab* happens to be 60°, and *am* 30°—but in this case the third index must not be finally put in equal to zero!

2. COMPUTATION OF ZONAL ANGLES BY THE MULTIPLE TANGENT TABLE.—The more important zones of the rhombohedral system are rectangular, and the multiple tangent table has therefore a wide sphere of usefulness, at any rate in the province of mineralogy. An inspection of a *stereographic* projection is sufficient to show that all zones passing through the poles $c(111)$, $a(10\bar{1})$ and $m(11\bar{2})$ are rectangular; and, with the isolated exception of the vertical zone, a *gnomonic* projection immediately indicates the correct co-efficient p/q. Thus, Fig. 65 shows that the zone 001 : 100 is rectangular and that the pole $e(101)$ lies at the summit of the zone. Correspondingly, Fig. 66 shows that the linear distance $ET = 3ER$ [where R should be pencilled in, as being the gnomonic pole of $r(100)$]. Whence it follows (no matter what the indices may be) that tan $ET = 3$ tan ER. In precisely the same way it follows that tan $(111 : 1\bar{1}1) = 4$ tan $(111 : 101)$; and so on. It might therefore appear that the investigator of a complex rhombohedral crystal is altogether independent of the Millerian formula; but this is not quite true as the single-circle worker may occasionally have to measure and subsequently develop an oblique zone, and he must naturally fall back on the cotangent form of the general Millerian formula. With regard to the vertical zone there are alternative methods of procedure. The first, and probably less direct, is to transform rhombohedral indices (hkl) into hexagonal indices $(pqrs)$ by the following formulae, which hold for all faces (not merely the vertical):—

$$p = -k + l; \quad q = h - l; \quad r = -(p + q); \quad s = h + k + l,$$

and look up 'Dana' for the theoretical values. The second and possibly quicker method is to use the multiple tangent table; for although, in absence of faces having the pinacoidal type of indices, it it not easy to deduce the co-efficient p/q by inspection, the routine method through the general Millerian formula is quite simple. Thus, given Miller's form $h(3\bar{1}\bar{2})$. To compute the theoretical angular value $(10\bar{1} : 3\bar{1}\bar{2})$, or in other words the phi-value of h. We choose our four poles $A(10\bar{1})$, $B(3\bar{1}\bar{2})$, $C(1\bar{1}0)$, $D(1\bar{2}1)$, so that $AD = 90°$. Then cross-multiplying indices we find that the co-efficient p/q of the *general* cotangent formula has the value $1/5$. Hence the formula $p(\cot AB) - q(\cot AC) = (p - q)\cot AD$ (in which the term involving AD vanishes since cot $90° = 0$) becomes: cot $AB = 5$ cot AC; or tan $AB = 1/5$ (tan AC) $= 1/5$ (tan 60°) $= 19° 6'$ by the multiple tangent table.

8. **The Converse Use of the Table.**—As mentioned in Section 2 (p. 69), the table can be used as a substitute for other methods in the preliminary determination of indices. If, for example, a projection of

topaz has been made from single-circle measurements, and primary indices have been allotted, the only indices that remain to be determined are those of *y*(021), *i*(113), *u*(112), *l*(120) and *g*(130). It is now open to the researcher to postpone the determination of the indices until the elements have been computed. The advantage attending this procedure is that he now knows the theoretical angular values *cf* (001 : 011) = 43° 52′, *co* (001 : 111) = 64° 2′, and *am* (100 : 110) = 27° 54′; and he can now proceed to *determine the indices of* the remaining *faces simultaneously with the values of the computed angles*. To take the case of *y*: since the angle *cy* is greater than the angle *cf*, we naturally look up *cf* = 43° 52′ in the first column, and find the corresponding value in the second column to be 62° 31′ (measured value 62° 25′). The second column proves the face to be (021). Now take the zone *co*. The mean measured values were *ci* = 34° 11′, *cu* = 45° 35′, and *co* = 64° 2′ (fundamental angle). All faces in this zone must have indices of the form (*hhk*). In this case the fundamental (initial) angle is the greatest; it is therefore useless to consult the first column. We look up the value 64° 2′ in the second column and find: 45° 45′ ⟵ 64° 2′. This incidentally proves that *u* has the indices (112). We must now consult the next column in order to derive a still smaller angle. Turning to the third column we find: 34° 23′ ⟵ 53° 51′ ⟵ 64° 2′. This value of 34° 23′ proves *o* to be (113), the intermediate value 53° 51′ relating to a face (223) which has not yet been observed on topaz. With Dana's choice of parametral plane, this face would be (443). If we start from Dana's standard value *cu* (001 : 111) = 45° 35′ and from this value of the third column proceed to the fourth, we get 53° 41′ as the rho-angle of this possible face.

9. **Note on the Supplementary Multiple Tangent Table.**—So far as the determination of indices is concerned, the supplementary table of p. 149 is more serviceable than the main table—not so much, perhaps, because it allows of a direct determination of indices involving any numbers up to 9, but rather because it is a one-page table. Its use presupposes that angular values up to 75° have been previously rounded off to the nearest half-degree, and values above 75° to the nearest quarter-degree. Given *am* = 53° and *ax* = 67°: to deduce the indices of *x*. Since 53 (1st col.) gives 69½ (2nd col.), we see immediately that the desired co-efficient p/q is a little less than 2—say, 7/4 or 9/5. To verify the first of these two estimates we look up 53 (4th col.) and find 67 in the 7th column, and there is therefore no need to try 9/5. Since p/q is 7/4, it follows that *x* is (470).

10. **Note on the General Implication of the Table.**—The previous remark about the possible face (443) of Dana will, perhaps, serve to emphasise the general applicability of the table. If any measured (or computed) angular value be given in a rectangular zone, then any other angle follows by an inspection of the indices (or of a typical gnomonic projection in difficult cases), the inspection being particularly easy if the given angle refers to a simple parametral form. The question therefore arises whether a multiple tangent table (possibly including a 7-column) could usefully replace the supernumerary computed angular values, which are generally quoted in large books of reference. It is, perhaps, a question which should not wholly depend on considerations of the space gained or lost in each particular case, for the table is not only applicable to crystals which have already been investigated, but also to those of the future.

EXERCISES.

I. GENERAL MILLERIAN FORMULA.

Exercise 32.—Given $A(010)$, $B(130)$, $C(110)$, $D(100)$ and $AC = 58° 4'$, $AD = 87° 6'$. Compute AB.

Exercise 33.—Given $A(010)$, $B(110)$, $C(100)$, $D(1\bar{1}0)$ and $AB = 58° 4'$, $AC = 87° 6'$. Compute AD.

Exercise 34.—Given $A(010)$, $B(111)$, $C(1\bar{1}1)$, $D(1\bar{3}1)$ and $AB = 68° 51'$, $AC = 103° 28'$. Compute AD.

**Exercise* 35.—Given $A(\bar{2}63)$, $B(001)$, $C(1\bar{3}3)$, $D(4.\bar{1}\bar{2}.3)$ and $AB = 44° 37'$, $AC = 70° 13'$. Compute AD.

Exercise 36.—Given $A(010)$, $B(011)$, $C(012)$, $D(001)$ and the phi-values B (11° 39′) and D (80° 27′). Compute the phi-value of C.

Exercise 37.—Given $A(010)$, $B(021)$, $C(013)$, $D(001)$ and the phi-values B (22° 19½′) and D (80° 33′). Compute the phi-value of C.

Exercise 38.—Given $A(010)$, $B(111)$, $C(1\bar{1}1)$, $D(1\bar{3}1)$ and the phi-values B (64° 50′) and C (106° 13½′). Compute the phi-value of D.

**Exercise* 39.—Given $A(110)$, $B(111)$, $C(001)$, $D(\bar{1}\bar{1}1)$; the angle $010 : 110 = 58° 4'$, and the phi values of B (64° 50′) and C (80° 33′)—both reckoned in the standard way from $b(010)$. Compute the standard phi-value of D.

II. TYPICAL HARMONIC EXAMPLES.

Exercise 40.—Given $A(010)$, $B(110)$, $C(100)$, $D(1\bar{1}0)$ and $AB = 58° 4'$, $AC = 87° 6'$. Compute AD.

Exercise 41.—Given in a monoclinic crystal $A(100)$, $B(101)$, $C(001)$, $D(\bar{1}01)$ and $AB = 41° 57'$, $AC = 69° 47'$. Compute AD.

Exercise 42.—Given $A(010)$, $B(120)$, $C(110)$, $D(100)$ and $AC = 58° 4'$, $AD = 87° 6'$. Compute AB.

Exercise 43.—Given $A(010)$, $B(120)$, $C(100)$, $D(1\bar{2}0)$ and $AC = 87° 6'$, $AD = 137° 34'$. Compute AB.

Exercise 44.—Given in a monoclinic crystal $A(100)$, $B(201)$, $C(101)$, $D(001)$ and $AC = 41° 57'$, $AD = 69° 47'$. Compute AB.

Exercise 45.—Given in a monoclinic crystal $A(110)$, $B(111)$, $C(001)$, $D(\bar{1}\bar{1}1)$ and $AC = 73° 49'$, $AD = 121° 20'$. Compute AB.

**Exercise* 46.—Given in an anorthic crystal $A(010)$, $B(110)$, $C(100)$ with $AB = 53° 3'$, $AC = 79° 19'$, and also a series of poles $D(210)$, $E(2\bar{1}0)$, $F(1\bar{1}0)$, $G(1\bar{2}0)$. Compute by a sequence of harmonic operations the angles which each of the latter poles makes with A.

Exercise 47.—Given in an anorthic crystal $A(010)$, $B(011)$, $C(012)$, $D(001)$ and the phi-values, $B = 11° 39'$, $D = 80° 27'$. Compute the phi-value of C.

Exercise 48.—Given in an anorthic crystal $A(010)$, $B(011)$, $C(001)$, $D(0\bar{1}1)$ and the phi-values, $B = 15° 56'$, $C = 31° 54'$. Compute the phi-value of D.

Exercise 49.—Given in an anorthic crystal $A(010)$, $B(012)$, $C(001)$, $D(0\bar{2}1)$ and the phi-values, $B = 22° 19'$, $C = 80° 33'$. Compute the phi-value of D.

**Exercise* 50.—Given in an anorthic crystal $A(010)$, $B(111)$, $C(101)$; the phi-value $B = 57° 19\frac{1}{2}'$, $C = 104° 4'$. Compute by a sequence of harmonic operations the phi-values of $D(121)$, $E(1\bar{1}1)$, $F(1\bar{2}1)$ and $G(131)$.

**Exercise* 51.—Given in a monoclinic crystal $A(110)$, $B(111)$, $C(001)$, $D(\bar{1}\bar{1}1)$, and the standard phi-values, $A = 41° 31'$, $B = 52° 3'$, $C = 90° 0'$. Compute the standard phi-values of $D(\bar{1}\bar{1}1)$ and $E(112)$.

III. INTERZONAL OR COPLANAR ANGLES.

NOTE.—A preliminary sketch stereogram will be of great assistance in every case.

Exercise 52.—In a certain anorthic crystal a zone-sheaf A, B, C, D, radiating from the face-pole (100), is intercepted by a fifth zone in the face-poles $a(010)$, $b(011)$, $c(001)$, $d(0\bar{1}1)$ respectively. Given the interzonal angles $AB = 35° 37'$, $AD = 120° 10'$. Compute the interzonal angle AC (*i.e.* the supplement of α).

Exercise 53.—In a certain anorthic crystal a zone-sheaf A, B, C, D, radiating from the face-pole (010), is intercepted by a fifth zone in the face-poles $a(100)$, $b(101)$, $c(001)$, $d(\bar{1}01)$ respectively. Given the interzonal angles $AB = 41°42'$, $AD = 144°30'$. Compute the interzonal angle AC (*i.e.* the supplement of β).

Exercise 54.—In another anorthic crystal the zone-sheaf A, B, C, D, radiates from the face-pole (001) and intercepts the primitive in (010), (110), (100) and $(1\bar{1}0)$ respectively. Given the interzonal angles $AB = 50°37'$, $AD = 144°19'$. Compute the interzonal angle AC (*i.e.* the supplement of γ).

**Exercise* 55.—In a certain anorthic crystal a zone-sheaf A, B, C, D, radiating from the face-pole (010), is intercepted by a fifth zone in the face-poles $a(100)$, $b(201)$, $c(001)$, $d(\bar{2}03)$. Given the angles $AB = 22°30'$, $AC = 64°4'$. Compute the interzonal angle AD.

IV. RECTANGULAR CASES (MULTIPLE TANGENT TABLE).

Exercise 56.—Given in a cubic or tetragonal crystal the angle am $(100 : 110) = 45°$. Deduce the angles which $a(100)$ makes with the following faces: (120), (130), (140), (150), (210), (310), (410), (510).

Exercise 57.—Given in a cubic crystal the angle *co* (001 : 111) = 54°44′. Deduce the angles which *c*(001) makes with the following faces: (221), (331), (112), (113).

Exercise 58.—Given in an orthorhombic crystal *cq* (001 : 011) = 52°43′. Deduce the angles which *c*(001) makes with the following faces: (013), (012), (021), (031); also (023).

**Exercise* 59.—In the same crystal *cr* (001 : 101) = 58° 10′. Deduce the various angles *cx*, where *x* is successively, (205), (203), (405), (302).

**Exercise* 60.—In the same crystal deduce the various angles *cx* where *x* is successively (109), (108), (106), (308), (508), (23 . 0 . 24).

Exercise 61.—Given the angle *bq* (010 : 011) = 57° 36′ in a monoclinic crystal. Deduce the various angles *bx*, where *x* is successively (012), (021), (031); (023), (032).

Exercise 62.—Given the phi-value, *q*(011) = 14° 44′, in a certain monoclinic crystal. Deduce the phi-values of (012), (013), (014), (015), (016); (023), (029).

Exercise 63.—Given the phi-value, *o*(111) = 55° 4′, in a certain monoclinic crystal. Deduce the phi-values of (121), (131).

NOTE.—In the following examples 64—67, the correct co-efficient p/q can be read immediately from a gnomonic projection. In actual research work this projection will, of course, have been previously prepared in every individual case, but for the purpose of these exercises the projection of Fig. 62 (p. 74) will serve as a general guide for the monoclinic and orthorhombic systems, provided it be clearly remembered which zones are rectangular in each case.

**Exercise* 64.—In an orthorhombic crystal:

(1) Given *ro* (101 : 111) = 32° 57′. Deduce *rx* (101 : 212) and *ry* (101 : 121).

(2) Given *st* (102 : 122) = 38° 58′. Deduce *sv* (102 : 112).

(3 Given *ct* (001 : 122) = 45° 28′. Deduce *cw* (001 : 121).

(4) Given *uw* (021 : 121) = 24° 58′. Deduce *ug* (021 : 221).

**Exercise* 65.—If the crystal of the previous exercise had been monoclinic, which angles (if any) could have been computed by the multiple tangent table.

**Exercise* 66.—Given the phi-value *s*(322) = 67°7′ in a certain monoclinic crystal. Deduce the phi-values of *t*(312), *x*(342), *y*(362).

**Exercise* 67.—Given the phi-value, *p*($\bar{1}$11) = 302°24′, in a certain monoclinic crystal. Deduce the phi-values of *v*($\bar{2}$12) and *z*($\bar{1}$31).

CHAPTER VII.

THE FOUR RELATED PROJECTIONS AND THEIR CRYSTALLOGRAPHIC HISTORY.

'Bei zwei invertirten Gestalten sind die Zonenebenen der einen die Flächen der andern, und die Flächen der ersten die Zonenebenen der andern; die Zonenrichtungen, Kantenrichtungen der ersten sind die Flächenrichtungen der andern, und die Flächenrichtungen sind die Zonenrichtungen der andern, und umgekehrt.

F. E. NEUMANN (1823).

'Die graphische Methode des Hrn. Prof. Neumann hat im Allgemeinen bei dem gelehrten krystallographischen Publicum geringeren Eingang gefunden, als man von einer so grossen Erscheinung hätte erwarten sollen. . . . . Und dennoch sehen wir in den verschiedenen krystallographischen Lehrbüchern ihrer kaum erwähnt!'

A. QUENSTEDT (1835).

So far, only two projections have been considered—the gnomonic and the stereographic. There are, of course, many other kinds of projection, each having peculiar advantages in the solution of particular problems. It must, however, be observed that the comprehensive study of projection belongs to one circle of activity, and the work of the laboratory to another; and as a pair of circles can only intersect in two points, there is obviously no well defined place here for any third projection. But an exception may be allowed in this supplementary chapter, since no crystallographic account of the gnomonic and stereographic projections would be complete without an allusion to their affinities, the 'linear' and the 'cyclographic.' Geometrically, the four projections are inseparably united. Crystallographically they have had an eccentric history.[23]

1. **The Four Related Projections.**—As already explained, the gnomonic projection, G, of a crystal face (see Fig. 67) is the point at which the *normal* to the face, drawn from a centre O' meets E the plane of projection. Now if the crystal face be moved parallel to itself, until it passes through the centre, O', it will intersect the plane of projection as a *line L, the linear projection of the face.* Since the angle $lO'G = 90°$, and since the plane $lO'G$ is perpendicular to the plane, E, it is possible to derive the linear projection from the gnomonic in the

[23] The four projections are described very fully: (1) by E. S. Fedorov (*Mining Journal* [Russian], 1887; and in the various editions of his text-book of crystallography and his treatise *The New* [*i.e.* Modern Projective] *Geometry* [Russian], 1907); and (2) by V. Goldschmidt (*Ueber Projection und graphische Krystallberechnung*, 1887).

following simple way. In the gnomonic projection (Fig. 68) place the main edge '*A*' of the crystallographic protractor along the diameter

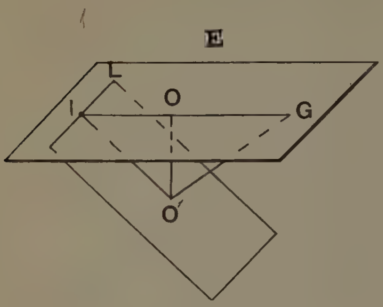

FIG. 67.

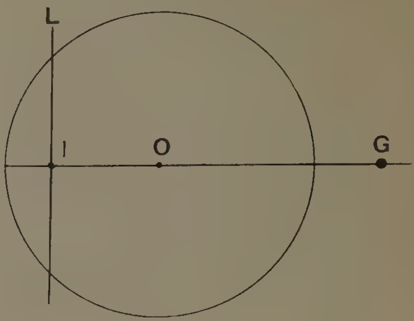

FIG. 68.

GO, and mark a point *l* at a gnomonic distance of 90° from *G*. The line *L* drawn through *l* at right angles to *OG* is the linear projection required.

The above is the modern form of the linear projection, *i.e.* Fedorov's 'linear' and Goldschmidt's 'euthygraphic.' If the projection plane, *E*, be supposed to be located below the centre of the crystal, then the crystal plane will be projected as a line, *L'*, on the same side of *O* as *G*. This is the Quenstedt form of the linear projection, as described in many text-books of crystallography.

It will be clear that in the linear projection a plane is projected as a line (of indefinite length) and a zone or edge as a point—the common point of intersection of the component face-lines. As a general instrument, the linear projection is inferior to the gnomonic, a group of projected face-lines being much more confusing than a corresponding assemblage of face-normals projected as points.

The relationship of the fourth projection (the 'Kugelprojection' of Quenstedt, 'gramma-stereographic' of Fedorov and 'cyclographic' of Goldschmidt) is not easily illustrated in terms of a general perspective figure. It has, however, precisely the same relation to the ordinary stereographic, as has the linear to the gnomonic. If a crystal face be not replaced by its normal, but merely translated till it passes through the centre of a sphere, it will intersect the sphere in a great circle, which is then projected on to the equatorial plane in the usual way. The relationship to the stereographic is illustrated by Fig. 69, in which *s* is the stereographic and *K* the cyclographic projection of a face; *s* is the pole of the circle *K*; in other words, the angular distance *Ks* is 90° stereographically. As in the linear, so in the cyclographic projection a zone is projected as a point: the common point of intersection of the various face-circles.

The relations of the four projections may now be illustrated by the help of the composite Fig. 70, in which s is the stereographic, G the gnomonic, K the cyclographic and L the (modern) linear projection of a crystal face; O the centre of the primitive and AA' the diameter, normal to Os (or stated otherwise the diameter parallel to L). Then s is the pole of the circle K, and G its centre; the plane angle $lAG =$

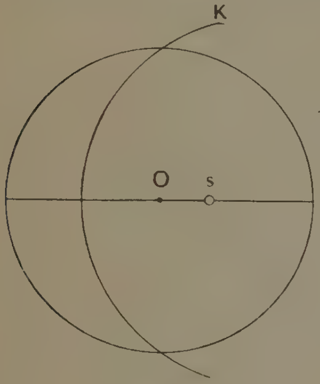

FIG. 69.

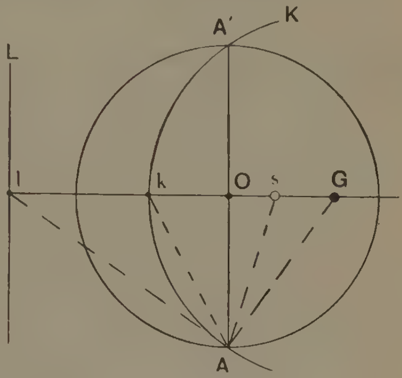

FIG. 70.

$90°$, $kAs = 45°$. Conversely, if L represents a gnomonic zone, then K represents the same zone stereographically projected; G is simultaneously the linear projection and the angle-point of L, and s its cyclographic projection, and so on. In view of these relationships, it is easy to pass from any one projection to any other by means of the main scale '*A*' of the crystallographic protractor.

2. **Their Crystallographic History.**—A full account of the earliest history of the stereographic and gnomonic projections is included in A. Hutchinson's paper (*Min. Mag.*, 1908, **15**, 93) describing his protractor. The present account may accordingly be suitably restricted to the special province concerned with the applications of all four projections to the problems of crystallography. Considerations of space rule out any extended treatment or the inclusion of anything in the nature of a bibliography.

All four projections were introduced into crystallography in 1823 by F. E. Neumann in the exceedingly important book, *Beitraege zur Krystallonomie,* an annotated edition of which has been recently issued by his son, C. Neumann (*Abh. K. Sächs. Ges. Wiss. Math.—Phys. Klasse,* 1917, No. 3, pp. 195—458). Realising that the gnomonic and stereographic projections were of greater utility, Neumann concentrated on these two modes of his 'graphical method' and relegated the 'linear' and 'cyclographic' variants to a paragraph at the end of his book. He pointed out the simple multiple relations between

the plane co-ordinates of the various faces, and inserted these plane co-ordinates on many of his plates, which included projections of the cubic system on $c(001)$, $o(111)$ and $d(101)$; idocrase, barytes, topaz, calcite, proustite-pyrargyrite and orthoclase. His work does not seem to have attracted much attention at the outset, but the stereographic projection (also treated by Neumann in a separate paper *Pogg, Ann.*, 1825, **4**, 63) was eventually taken up and adapted to a remarkably simple and symmetrical form of logarithmic treatment by Miller, with results that are familiar to all. The gnomonic, on the other hand, was not only neglected, but temporarily eclipsed by A. Quenstedt's development of its 'Polar form'—the linear projection (*Pogg. Ann.*, 1835, **34**, 503, 651; **36**, 245, 379).

The gnomonic projection was the subject of a remarkable paper by W. H. Miller (*Phil. Mag.*, 1859, Ser. 4, **18**, 37), who investigated some of its properties but apparently made no attempt to introduce it into the body of crystallographic practice. The credit of being the first to appreciate this projection at its true value would appear to belong to E. Mallard, who not only used the projection concurrently with the stereographic in his text-book (*Traité de Cristallographie, Tome* 1, 1879, with an Atlas of exquisite plates), but also established the important relation (*Traité*, pp. 1—65, especially pp. 26, 63) between the gnomonic projection and the *polar* lattice of Bravais (not the 'ordinary' Bravais lattice), which allows of a graphical comparison of reticular densities, or of their reciprocals, the X-ray physicists' atomic grating-distances.

But although the more fundamental properties of the stereographic and gnomonic projections were thus exhaustively investigated, much remained to be done in the matter of their practical application. Apparently this was first realised by Fedorov (who developed the technique of graphical methods in conjunction with two-circle measurements of geometrical and optical constants), and independently by Goldschmidt (whose two-circle method of goniometry has been widely adopted by mineralogists). Penfield's contributions, though more conservative, were perhaps none the less important, since they revealed what can be done with the stereographic projection under the relatively unfavourable circumstances of single-circle gonimetry. For an imperfect list of other workers a reader may well consult the Index.

CHAPTER VIII.

A NEW SYSTEM OF PRACTICE.

'Of the eight angles given by this author, only two are correct: it may therefore be taken for granted that what he has given are not actual measurements, especially since they are given to a second, which no goniometer will indicate, but rather assumptions founded on the known laws of decrements belonging to the cube, which he erroneously assumed as the primary form. . . . It is known that the natural planes even of small crystals, which, being the most perfect, are best adapted to the use of the reflective goniometer, do not commonly give coinciding measurements, but are subject to variations from 1 to about 40 minutes.' W. PHILLIPS (1821).

'J'ai cru aussi qu'il devenait désormais indispensable de placer, en regard des incidences calculées au moyen d'un petit nombre de données fondamentales, les incidences mesurées directement, avec l'indication du nom de chaque observateur. La plupart des monographies publiées depuis une vingtaine d'années nous ont en effet revélé l'existence de formes dérivées dont la position, assurée par les zones auxquelles elles appartiennent ou par des mesures prises avec soin, ne peut pas s'exprimer en fonction des arêtes de la forme primitive par des nombres aussi simples que ceux auxquels Haüy avait cru devoir s'arrêter. Peut-être parviendra-t-on un jour, par la comparaison des angles calculés et des angles mesurés, à fixer la limite des erreurs dues à l'observation elle-même ou à l'imperfection des cristaux et à établir, moins arbitrairement qu'on ne l'a fait jusqu'ici, où s'arrêtent les rapports simples et où commencent les rapports compliqués.' A. DES CLOIZEAUX (1862).

'Probably all crystallographers have found numerical calculations burdensome, and the only real way to absolutely avoid mistakes is either to do all work in duplicate, or to apply some other method of checking results.' S. L. PENFIELD (1902).

'The application of crystallography to the identification of chemical substances appears to me to be so important that every hindrance to its development must be eliminated' [Russian]. E. S. FEDOROV (1913).

'Frontiers have always been the scenes of greatest activity, and it is precisely on the borderland between two sciences that the most fruitful progress has been made.' H. A. MIERS (1918).

An attempt has been made in the preceding pages to present a judicious selection of crystallographic methods—some very old, others quite new—which experience has proved to be accurate and time-saving. Unless one or two better methods have been inadvertently omitted (a not unlikely possibility in view of the somewhat inaccessible nature of some of the original literature), it must be concluded that no further improvement can be expected unless one is prepared to make a critical survey of geometrical investigation as a whole, with the eventual object of eliminating everything but essentials. The exploration of every possible way of exercising economy of effort would seem to be especially needed at the present moment in the history of crystallography. It is a need that is no doubt particularly felt by the researcher who devotes himself to the study of laboratory products, confronted, as he is, with some tens of thousands of substances which have not yet been measured. Now the same purely scientific considerations which have prompted the investigation of 'mineral' or 'organic' crystals in the past make it desirable that every one of these new compounds should be investigated. Moreover, the inducement to measure every possible substance is certainly not lessened by the fact that a crystallographic description of any laboratory product has

recently acquired a practical value, far surpassing that which it has long enjoyed in the province of mineralogy. This somewhat unexpected advance is due to Fedorov, who has not only worked out the necessary principles, but has actually effected a classification of the existing body of geometrical data into an orderly system, so that it is now possible to identify any one of these substances by virtue of its characteristic crystalline form. Although some ten thousand substances have already been classified, the addition of at least an equal number will be required if 'crystallochemical analysis' is to become generally applicable to the wide range of organic chemistry. It is, of course, impossible to enter any compound into the lexicon, and thus make its identification a future possibility, unless its geometrical characters have been previously determined.

The publication of this lexicon, 'The Crystal Kingdom,' undertaken several years ago by the Petrograd Academy of Science, was unfortunately delayed by the outbreak of war. The book was in the press in the year 1917, but has not yet appeared, the second delay being presumably attributable to the outbreak of civil war in Russia. Whether the untimely death of Professor Fedorov will affect the ultimate publication in Russia, it is of course not possible to say. Fortunately, at least one complete set of proof sheets exists, the possession of which made it possible to identify a crystal of salol (phenyl salicylate) on a recent occasion (T. V. Barker, *Lancet*, 1917, 798).

The present chapter is accordingly devoted to a general examination of the course of a crystallographic investigation, with the object of determining as far as possible what is indispensable, and how far these indispensable operations should be carried. In this connexion it will be useful to have in view a concrete description, and a fairly typical investigation of an anorthic compound (that of dibromoinosite tetracetate, *J. Chem. Soc.*, 1907, **91**, 1784; Groth's *Chemische Krystallographie*, **3**, 610) will be selected. It may here be noted that the horizontal plan, accompanying the original description, is not true to scale, but as the description differs in no other way from the numerous descriptions with which a reader must be familiar, it does not seem necessary to reprint it in full (an amended description will be given at the end of this chapter). The various items in this typical presentation of single-circle goniometry will now be considered, with the exception of those which do not seem to call for any comment; and it need scarcely be added that what is true for single-circle goniometry also holds in the main for two- (and its variant three-) circle work (the necessary modifications will be considered later). The order of treatment is that generally adopted in a published description, and not that of the actual work on which it is based.

1. **The Elements.**—The office of 'elements' is to present an epitome of the geometrical characters of the crystal, a task which they are well able to perform, provided they are accompanied by a list of forms in terms of Millerian (or other generally recognised form of) indices—a most important condition, for *elements pure and simple,*

afford no definite information concerning any crystal. They can be held to fulfil this important duty satisfactorily, provided they obey the following three conditions: (1) they must have been computed by methods which are at least as accurate as the original measurements on which they are based—which implies a logarithmic or some other exact form of computation, since it is extremely difficult, if not impossible, to attain such a degree of accuracy by graphical methods; (2) they must be free from arithmetical mistakes; (3) they must be given in a useful form, *i.e.* a form which will enable a future worker to reconstruct the crystal, either in projection or through exact calculations, with a minimum expenditure of effort. With the conspicuous exception of the anorthic system, the orthodox axial-ratio form would seem to meet every reasonable requirement.

In the case of the orthorhombic system for example, the ratios $a:b$ and $c:b$ respectively represent the tangents of the angles *am* (100 : 110) and *cq* (001 : 011), and a table of natural tangents (see Appendix III) therefore admits of a ready conversion into a form which is adapted to any subsequent development, logarithmic or graphical. The reconstruction of a monoclinic crystal is only one step less easy (*cf.* Fig. 71). The calculation of *am* only involves the solution of a right-angled triangle in which $(180° - \beta) = ca$ is a side and c an angle with a tangent equal to $a:b$ ($\tan am = \sin ac \times a:b$). Moreover, the angle *cq* (001 : 011) is deduced by the analogous formula $\tan cq = \sin ac \times c:b$. Any further angles in the zones *ab* and *cb* are read from the multiple tangent table, but any further form development in the zones *ac* and *mc* naturally demands the solution of an oblique triangle as a preliminary. Thus, the angles *ar* and *rc* (where *r* represents the face 101) are obtained simultaneously from a plane triangle involving the axial lengths a and c and the included angle β. The right-angled triangles *cro* and *cam* are then solved for *co* and *cm* respectively. The non-rectangular zones *arc* and *moc* can then be developed (if necessary) by the cotangent form of the Millerian formula.

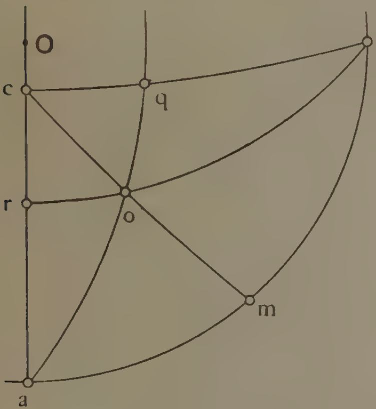

FIG. 71.

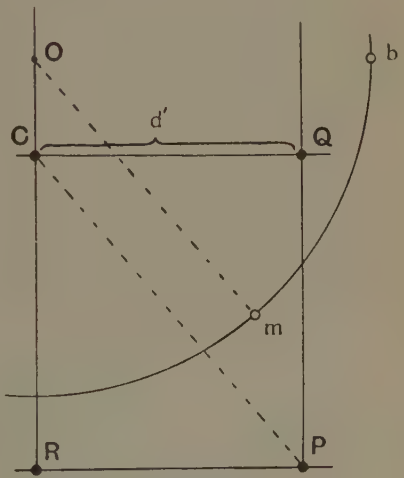

FIG. 72.

[NOTE.—In what follows, the letter p (gnomonic equivalent P) has been substituted for o as the designation for (111)]. With regard to projection, the elementary gnomonogram can be established by the following simple operations: (1) the angular distance (*cf.* Fig. 72) of the gnomonic pole of C (00) is given by the angle $(\beta - 90°)$, a horizontal line drawn through which point represents the zone-line *cb*; (2) the

angular distance d' of the perpendicular zone-line QP, being known from the relation tan d' = axial ratio $c : b$, we have no difficulty in drawing this zone-line QP, and the point of intersection of the two lines locates the gnomonic pole of Q (01); (3) the unit distance QP is then determined as a point of intersection of QP with a line, obtained by translating the radius Om till it passes through the origin of co-ordinates C (00); (4) the elementary gnomonogram $CQPR$ can then be completed by drawing PR parallel to QC, and subsequently reduplicated as required.

There are several *purely* graphical ways of preparing a gnomonic or stereographic projection from the elements of a monoclinic crystal (one devised by the present writer need scarcely be published, as it appears to be no better than the others). Experience indicates that accuracy demands a previous calculation of the angle *am* from a right-angled triangle. The above method, which is really simpler than the description would suggest, has been given in full in order to support a conclusion, derived from an extensive practical study of all forms of elements, that the axial-ratio form fulfills all reasonable requirements in every system except the anorthic. This conclusion, it is fair to add, is not in agreement with the expressed views either of Professor Goldschmidt or of E. S. Fedorov, each of whom has suggested novel forms of elements (peculiarly advantageous for certain purposes) and recommended their adoption in all the systems. As mentioned on a previous page, an important feature, peculiar to the Goldschmidt 'polar elements,' is that they are adapted to the derivation of very accurate constants, for all the measured angles can be taken into account. Whilst expressing the view that there is no room in crystallography for slip-shod work, the present writer believes that the attainment of such exceptional accuracy is not at present demanded in the study of laboratory products in view of: (1) the immense numbers of such compounds, (2) the small numbers of those who are actively interested, and (3) the fact that the difference between 'very accurate' and 'accurate' values is of the order 5'. From this practical point of view the Fedorov 'angular elements' would seem to be preferable because, although only representing 'accurate' results, they are more in accordance with classical crystallography both in form and in mode of derivation. They are therefore commended here to the attention of single-circle workers in the isolated case of the anorthic system. It is unfortunate that the fullest advantages of both Goldschmidt and Fedorov elements can only be made by two-circle workers, but as will be seen presently the single-circle worker should feel quite 'at home' with the Fedorov elements; which by the way should not be confused with the 'projectivity elements'—perhaps the least happy crystallographic suggestion ever made by this author.

In the anorthic system the axial-ratio form of elements is quite unsuitable. Originally prompted by theoretical rather than practical considerations, they happen in this system to constitute not so much a *cul-de-sac* as an actual trap, from which it is most difficult to escape. It is of course a fairly easy matter to calculate the anorthic elements $a : b : c$, α, β, γ, from a given set of angles, but it is an exceedingly tedious, not to say difficult problem to effect the reverse transformation —a conclusion that a properly sceptical reader can test by deriving the formulae and then applying them to a concrete case. Graphical solutions have from time to time been offered, but a really practicable method (*i.e.* one which is both accurate and rapidly carried out) has yet to be discovered. It is interesting to note in this connection that axial ratios are a comparatively modern feature in mineralogical treatises. They were not used, for example, in Brooke and Miller's celebrated edition (1852) of Phillip's *Mineralogy*. Nor were they employed in the first three editions of Dana's *Mineralogy* (a fact which has been kindly pointed out to me by Dr. L. J. Spencer). All these authors quote the necessary number of fundamental *interfacial* angles as elements in all the systems. A general reversion to this custom would certainly bring about an improvement, especially in the anorthic

system. Still more preferable, perhaps, would be a general adoption (in the particular case of the anorthic system) of Fedorov's form of angular elements (*cf.* Fig. 73), *d, e, ab, f, g*, where *ab* represents the *interfacial* angle 100 : 010, and *d, e, f, g* are the *interzonal* angles marked in the Figure. Such elements are eminently suitable both for the logarithmic calculation of angles and the preparation of an accurate projection. Where calculations are concerned, it is possible to develop the interzonal angles at the poles *a*(100) and *b*(010) by means of the cotangent form of the Millerian formula, a knowledge of which angles, together with the side *ab*, admits of the solution of a series of spherical triangles. The angular elements also lend themselves admirably to the preparation of a projection. The elementary gnomonogram, for example, is readily constructed as follows (*cf.* Fig. 74). The stereographic poles *a*(100) and *b*(010) are marked on the primitive from the known angle *ab*, and the diameters *Oa*, *Ob* are then drawn. Along a perpendicular to *Oa* two points e', d', are located at *gnomonic* distances $Oe' = 90° - e$, $Od' = 90° - d$ respectively. Similarly, points g', f' are located along a perpendicular to *Ob* at *gnomonic* distances $Og' = 90° - g$, $Of' = 90° - f$. Lines drawn through e', d', parallel to *Oa*, and through g', f', parallel to *Ob*, define by their mutual intersections the elementary gnomonogram, (00) — (01) — (11) — (10).

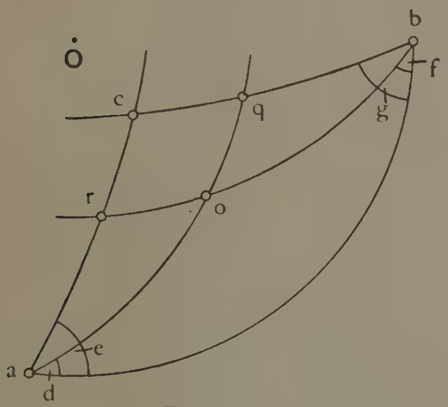

FIG. 73.

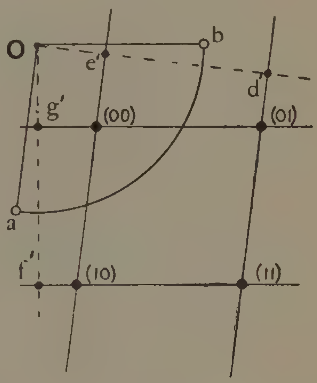

FIG. 74.

It will be realised that $\alpha = (180° - e)$ and $\beta = (180° - g)$; also that the angles *d, e, f, g* are generally involved in the calculation of the axial-ratio form of elements. The Fedorov angular elements are therefore closely related to the orthodox form. The practical disadvantage attending the use of the latter form is mainly due to the fact that the component parts of α and β are omitted. If these parts (*i.e.* the angles *d* and *f*) were added as a kind of appendix, the orthodox elements

would cease to be primarily ornamental in the particular case of the anorthic system; but they would still be somewhat unsatisfactory, since the preparation of an accurate projection would involve as a preliminary the computation of the interfacial angle *ab* from an *oblique* spherical triangle.

2. **Forms and their Symbolisation.**—Several not unimportant points of detail arise under this general head. They will be dealt with separately.

CHOICE OF AXES AND PARAMETRAL PLANE.—An original author has generally a considerable freedom of choice in both these particulars. It will however be generally agreed that an attempt should always be made to attain simple indices, if only to lessen the probability of a subsequent worker feeling himself called upon to introduce a revised orientation. It need scarcely be added that the usual conventions should be strictly adhered to in the common interest. The allocation of any other indices save (010) to the symmetry-plane of a monoclinic crystal, for example, is liable to create a future confusion to an extent which is only equalled by the frequent changes of setting of a former era.

DOUBTFUL FORMS.—Recent developments in crystallography emphasise the necessity of caution in recognising 'doubtful forms,' for the somewhat easy admittances of past observers have made it incumbent on present-day workers to effect a general revision. In the typical case of the inosite compound under review, two 'forms' having indices $(\bar{7}04)$ and $(\bar{5} \cdot \bar{7} \cdot 12)$ were cited. As these two planes were not represented as parallel pairs of faces, and were only observed on one crystal, their publication served no useful purpose. The inclusion of a doubtful form obviously detracts from the general value of a description, and merely serves to overload books of reference with questionable material. Whether a form is doubtful or no must depend on the observer's discretion. Any form which is not common to two simple zones should always be regarded with suspicion and not accepted unless it occurs on more than one crystal of the same crop. A form $(\bar{7}04)$, for example, only lies in one simple zone, namely, [010].

INDICES.—As is well known, Weiss' intercepts and Lévy's and Naumann's indices have been generally replaced by Miller's indices, largely owing, no doubt, to the extraordinary adaptability of the latter to processes of computation. Their only rival at present is the Neumann gnomonic system of plane co-ordinates which has been recently adopted by Goldschmidt and amplified by the invention of two-figure indices for the vertical faces. A dual citation of the Millerian and Neumann-Goldschmidt symbols is unnecessary in view of the ease with which they are mutually transformable; thus, for a terminal face, Miller $(hkl) \rightleftarrows h/l,\ k/l$ Neumann. And, as the Neumann plane co-ordinates are but a special case of the general three dimensional Mil-

lerian indices, it would seem that of the two the Millerian indices deserve to be universally adopted for purposes of publication, the transition to the Neumann-Goldschmidt form being effected mentally as required.

SIGNIFICANT LETTERS.—The most systematic selection of letters is naturally that of Lévy, but it has never been generally accepted outside France. In other countries letters are solely employed as a convenient aid in referring to an annexed diagram, but even in this relatively humble sphere there is obviously room for a systematic choice. In fact, if a definite allocation of letters to the common forms be generally agreed upon (it does not particularly matter which letters), the citation of indices becomes unnecessary. Two principles would necessarily appear to govern the choice of letters: (1) avoidance of Greek letters is desirable in this era of the typewriter in the laboratory and linotype machine in the composing room; (2) the system should not be carried so far as to tax the memory or to exhaust the whole of the alphabet for stereotyped purposes. These considerations have been kept in view in drafting the following suggestions, which, it will be realised, are largely founded on schemes adopted by Professor Groth, the essential difference being that capital italics are substituted for small Greek letters.

ORTHORHOMBIC SYSTEM.—It is desirable to provide for $a(100)$, $b(010)$, $c(001)$; then $m(110)$, $q(011)$, $r(011)$; also $o(111)$; and, finally, $l(210)$, $n(120)$. This would seem to be a suitable range. A stereotyped usage is suggested for the last two forms, because the vertical zone is more likely to exhibit rare forms, being generally selected for that very reason. If the correspondence between the alphabetical sequence l, m, n, and the geometrical sequence a — (210) — (110) — (120) — b be noted, there should be no tax on the memory. The above list is sufficiently exhaustive to cover the form development of the great majority of orthorhombic laboratory products. Any further forms might well be given any letter whatsoever *ad hoc*.

Example.—Suppose topaz were a laboratory product, and had been found by an original investigation to exhibit the forms of Figs. 58—59, p. 70, which were there described as $m(110$, $l(120)$, $g(130)$, $b(010)$, $y(021)$, $f(011)$, $c(001)$, $i(113)$, $u(112)$, $o(111)$, $d(101)$; then the description could well be abbreviated in any abstract or book of reference (or even in an original paper) to $mng(130)by(021)qci(113)u(112)or$—in which a few letters have been substituted for others in accordance with the presupposed, generally accepted scheme.

MONOCLINIC SYSTEM.—The only requisite modifications are a distinction between $r(101)$ and $R(\bar{1}01)$, and between $o(111)$ and $p(\bar{1}11)$. The various forms provided in this, the most important system, are given in Fig. 75.

ANORTHIC SYSTEM.—Several further distinctions have now to be made —wholly by a further recourse to capital letters. The final result is given in Fig. 76.

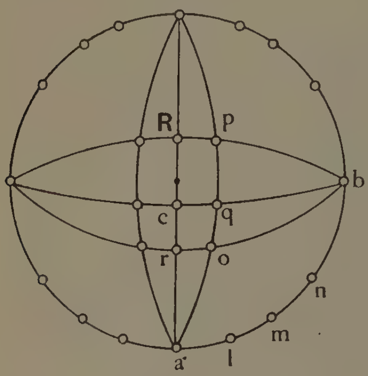

FIG. 75.

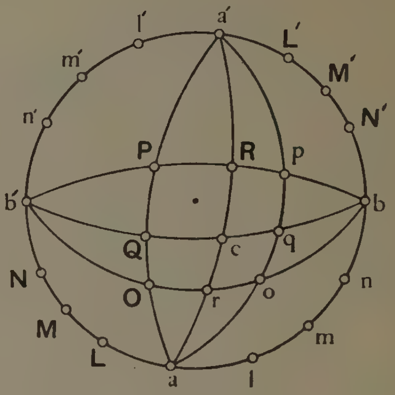

FIG. 76.

Examples.—With the exception of two doubtful forms which are here deleted, the list given in the original description of the inosite compound is as follows: $b(010)$, $m(110)$, $a(100)$, $n(1\bar{1}0)$, $o(011)$, $c(001)$, $q(0\bar{1}1)$, $k(\bar{1}\bar{1}1)$, $d(\bar{1}01)$, and, rarely, $p(\bar{1}11)$. The amended list now simply becomes: *bmaMqcQPR,* rarely *p*. Similarly the fairly lengthy list of forms which have been observed at different times on copper sulphate becomes $bmaLMNY(1\bar{3}0)$ $t(021)$ $qcQTz(1\bar{3}1)$, $x(\bar{1}21)$ $pXF(1\bar{3}1)$. Attention may be called to the use of *Y* as opposed to *y*, for example, as indicating that *Y* lies in the left half of a projection, also to the fact that the indices of *T* and *X* need not be introduced, since the specific meaning of *t* and *x* have been previously defined.

So much for the relatively important systems. The four highly symmetrical systems require very little notice.

CUBIC SYSTEM.—Cube, dodecahedron, octahedron *a, d, o.*

TETRAGONAL SYSTEM.—$a(100)$, $l(210)$, $m(110)$, $c(001)$, $r(101)$, $o(111)$.

HEXAGONAL SYSTEM.—$c(0001)$, $m(10\bar{1}0)$, $o(10\bar{1}1)$ and possibly $a(11\bar{2}0)$.

RHOMBOHEDRAL SYSTEM.—Following the system, generally used in the case of calcite, we have $c(111)$, $r(100)$, $a(10\bar{1})$, and, less frequently in the province of laboratory products, $e(110)$, $f(11\bar{1})$, $b(2\bar{1}\bar{1})$ and $v(20\bar{1})$.

LESS SYMMETRICAL CLASSES.—These rarely discover themselves to an observer in the province of laboratory products, and it therefore does not seem to be necessary to work out any detailed scheme.

THE PARALLEL FACE.—Whenever it is desirable to distinguish between a face and its parallel, the use of the apostrophe sign is sufficient. Thus, $a(100)$ and $a'(\bar{1}00)$.

3. **Number of Crystals Measured.**—In view of the present-day objects of crystal measurement, it would seem that too many crystals

of a given substance are subjected to measurement. Granted fair material, the mean results derived from, say, two or three crystals are extremely close to those derivable from five or ten. A general practice of measuring two (or, if occasion requires it, three) crystals would probably meet all requirements, provided some ten others were inspected with a lens in order to ensure the representative character of the crystals actually measured.

4. **Number of Zones Measured.**—The highly prominent zonal development of a crystal makes it possible to fix the relative positions of all its faces in terms of a surprisingly small number of zones. In the case under review four zone measurements would have been sufficient, but sixteen were actually carried out, mainly, no doubt, with the object of confirming the system. But the latter aim could have been equally well fulfilled by the quantitative study of a stereogram, prepared from the irreducible minimum of four zone measurements.

It seems desirable at this stage to call attention to the inevitable penalties of over-measurement, as being typical of the results of over-indulgence in many other crystallographic operations. No record was kept at the time of the number of hours devoted to measurement, but it must have been equivalent to three working days of eight hours each (twelve crystals were wholly or partly measured). Compared with this is the half-a-day that would have amply sufficed for a prudent measurement of two crystals (or the two hours, if a two-circle goniometer had been employed). The sort of question which has to be decided by every researcher is whether the additional result (in this case, a possible further approach to truth, amounting to $5'$ at the outside) repays the extra work.

5. **Citation of Number of Angles Measured.**—This is usually effected by means of a special column. A preliminary statement of the number of crystals measured, accompanied by a specification of the occasional forms, would seem to meet all reasonable requirements.

6. **Citation of Limits.**—In view of the extensive space demanded by a table of limits (amounting to twice the space required by the most important part of the table—the mean measured values), it is suggested that the publication of limits be discontinued and that a 'query' (?) be appended to any particular mean result which appears to be especially untrustworthy on account of a wide fluctuation of the component angular values.

7. **Citation of Mean Measured Angles.**—In single-circle goniometry these are generally quoted in the form, AB, BC, CD, DE, EA', *i.e.* a form obtained by taking adjacent differences from the actual goniometer readings, but a citation in the form AB, AC, AD, AE (in which face A is supposed to refer to a face with the pinacoidal type of indices) would be more useful to a future worker, as facilitating the prepara-

tion of a projection and the application of the cotangent form of the Millerian formula (and of the multiple tangent table in the special case of rectangularity).

8. **Mutual Adjustment of Fundamental Angles.**—The minimum number of fundamental angles is generally made the basis for the computation of the elements. In this connection it may be pointed out that the multiple tangent table sometimes makes it possible to employ a greater number of angles than the minimum. Most topaz crystals, for example, exhibit three pyramidal forms lying in the zone *cm*, namely, *i*(113), *u*(112) and *o*(111). In the ordinary way one of these angles would be selected as a fundamental and the other two neglected (so far as the accuracy of the elements is concerned, they might have been entirely absent). All three values can, however, be readily taken into account by converting the mean *ci* and *cu* values into corresponding *co* values by means of the multiple tangent table. A simultaneous application of the table to a second zone will obviously finally lead to an enhanced value of the axial ratios. In any case in which this method is adopted, the finally adjusted values should naturally be quoted in some form or other. Perhaps the case would be met by substituting, say, a fundamental value $co = {}^*64°2'$ by $co = {}^{**}63°56'$ $(64°2')$, the double asterisks serving to call attention to the fact that it is an adjusted and therefore more trustworthy fundamental value, whilst the term enclosed in parentheses indicates the mean measured value for the angle in question. This method may possibly commend itself to a researcher who would otherwise feel it desirable to measure more crystals as a means of increasing the accuracy of his work. Cases are not infrequent in which three minutes' work with the table is as profitable as a further three hours' work with the goniometer, so far as the accuracy of the elements is concerned.

The principle of the above method is also applicable to the non-rectangular zone, but in this case the actual computations have to be effected by the individual worker. Thus, suppose a monoclinic or anorthic crystal develops the four tautozonal faces (or coplanar zone-axes) A, B, C, D; it is possible to utilise all three measured angles AB, AC and AD by provisionally adopting AB and AC as fundamentals and adjusting these values by the help of AD in the following way. The values AD and AB by the cotangent formula give a new value for AC; correspondingly, the values AD and AC give a new value for AB. We now take the mean of the actually measured and half-measured-half-computed values of AB and AC as the accepted fundamentals (and prefix double asterisks in the table). Any eventual faces E and F can be treated in the same way, but it ceases to be necessary to take a repeated account of the measured values AB and AC, in taking the final means, as they inevitably acquire a dominating influence by the

very process of computation. This feature of the method need scarcely lead to any undesirable results, since the three 'fundamental faces' A, B, C can be initially selected from the whole group A, B, C, D, E, F, as giving the three most sharply defined reflections.

The method of applying such simple methods of adjustment needs no further explanation in the case of an orthorhombic crystal. In the monoclinic system the utmost use should naturally be made of the multiple tangent table—whenever, for example, a zone *ba* (*bc* or *br*) exhibits faces additional to the unit prism or dome. Whenever two such zones are developed, two out of the three necessary fundamental angles of enhanced accuracy are thereby obtained. Of much greater frequency, however, is a rich development in one or more non-rectangular zones such as *ac* and *cm* (*cf.*, for example, Marignac's description of the hexahydrated nickel sulphate in Groth's *Chem. Kryst.*, **2**, 425). In such a case two fundamentals are naturally selected from one of these zones (after a mutual adjustment by the cotangent formula), and the third, if possible, from an adjusted zone passing through *b* (010). In the anorthic system there are generally at least two well-developed zones; *cf.*, for example, copper vitriol, or, again, an isomorphous series (more suitable for practice in goniometry) of the tetra-oxalates variously investigated by Rammelsberg, Wyrouboff, de la Provostaye and Des Cloizeaux (*ibid.*, **3**, 140-143—as suggested by Professor Groth, the ammonium salt is at present under investigation). The application of the cotangent formula thus leads to four fundamentals of enhanced accuracy, leaving a fifth (necessarily taken from a third zone) to the discretion of the computer, who might give it a preliminary adjustment somewhat as follows. Suppose this fifth angle is the angle AB of a zone A, B, C, D, E. From AC and AD we compute AB; similarly from AD and AE. These two computed values of AB are averaged, and the mean taken between it and the measured value AB.

9. **The Computation of Theoretical Angular Values.**—At the outset it will be convenient to consider quite briefly the early history of the practice of computing theoretical angular values. These computations were, of course, involved in the discovery by Haüy of the Law of Simple Decrements (Law of Simple Indices or Zone Symbols or Multiple Intercepts), being in fact demanded by any exact proof of that law. The subsequent invention by Wollaston (1809) of a more accurate form of goniometer naturally led to a series of more rigorous tests—tests (*cf.* p. 91) which 'proved' the law as far as the nature of the material allows, for the art of measurement had already been carried to a point of refinement far exceeding the degree of constancy of the theoretically equal, angular values. The tables of angles in this early period (as, *e.g.* in the second edition of Phillips' *Mineralogy*, 1823) were a mixture of observed and computed angles, apparently no attempt being made to distinguish one kind from the other; but as time went on more and more weight was attached to computed values, the movement being no doubt accelerated by the simplification in methods of calculation introduced by Miller in 1839. Apparently the first Mineralogical treatise to adopt computed values exclusively (fundamentals, of course, excepted) was Dufrénoy's *Traité* (1844—5). The same course was taken by Brooke and Miller (1852), but des Cloizeaux, in his *Manuel de Minéralogie* (1862), felt it necessary to include also the observed values, 'in view of the circumstance that the theoretical values are generally computed from a small number of data.' Later mineralogical treatises have reverted to purely computed values. On the other hand, com-

pendia of the type of Rammelsberg and Groth, devoted to chemical substances in general, include both measured and computed values.

So much for the origin of the custom of computing theoretical angular values. The question now arises whether it continues to serve any useful purpose, theoretical or practical. The answer to the question does not appear to be in doubt. In the first place the original and apparently only theoretical rôle of the custom (the proof of a fundamental law) can scarcely be said to have any present-day interest —the law has been vindicated so many thousands of times in the past that no further proofs are called for. In the second place theoretical angles have never had any practical value which is not at least equally shared by the observed angles, for they are no more trustworthy on the whole than the observed values (having generally been derived from an arbitrarily selected part of the latter), and the differences between the two are so small that they can be interchanged for all practical purposes. Finally, so far as future potentialities are concerned: if at any future time the fundamental laws of geometrical crystallography were to demand a further consideration, any present-day discontinuance of calculation will have furnished the ideal material—a mass of unprejudiced, observed data.

The question now arises whether it is really possible to avoid computing theoretical angular values in the course of an accurate investigation. A little reflection will show that it is almost impossible in any system of crystallographic practice which ignores the existence of the gnomonic projection. Such a system is that which is (or was until recently) in general use, for so far as projection is concerned it depends solely on the stereographic projection—not so much as a quantitative instrument as a guide to the serious work of calculation. If indices are to be determined by formal processes of computation (by the solution of spherical triangles and by the Millerian formula), then the reverse calculation of theoretical angular values is almost inevitable; for the additional work involved in replacing the approximately simple co-efficient p/q by the truly simple co-efficient, implied by a rigorous subordination of the crystal to a law of simple indices, is comparatively trifling. The position is, however, completely reversed in the case of a worker who calls to his aid the gnomonic projection. Unless impelled by overpowering, theoretical or other considerations, he may well pause after determining his indices—an operation lasting a few minutes—to reflect whether (in the general case of an anorthic crystal) it is worth while to devote some hours or days to the solution of a series of triangles.

The next question is whether the combined gnomono-stereographic projection is reasonably trustworthy (no method is of course 'absolutely trustworthy'—the classical literature naturally contains many

mistakes). This, like many other questions discussed in this chapter, is a matter for personal trial. The present writer finds that the combined projection satisfies every reasonable requirement. In the first place, when properly used, the stereographic projection guards against errors (or such as are mutually inconsistent) in the original measurements; and it is therefore quite unnecessary to calculate theoretical angular values in order to check the measured values. In fact, the author's experience is that the numerical solution of every spherical triangle must be checked graphically, if the ten per cent. (or thereabouts) of wrong solutions are to be rectified before they can exert further inconvenience. In his case, then, it is not a case of checking the measurements or graphical work by logarithmic computations, but exactly the opposite. The second, and only remaining, point refers to the accuracy of the actual determination of indices by means of the gnomonic projection. In this connexion it only seems necessary to refer a reader to the earlier part of this book with the numerous diagrams, and also to remind him that for the greater part they are based on the relatively unfavourable data of single-circle goniometry. Now and then (it may be only once in the life-time of a worker in the province of laboratory products) there are cases in which the gnomonic diagram loses its usually conclusive value; but as such cases can be recognised at the outset (after a little experience with the projection), there does not seem to be room for any anxiety in this direction (the case discussed on p. 40 appears to be especially instructive).

A further consideration of 'doubtful forms' which generally means 'forms with complex indices' may be of interest. The doubtful face x (-5, -7, 12), observed on a

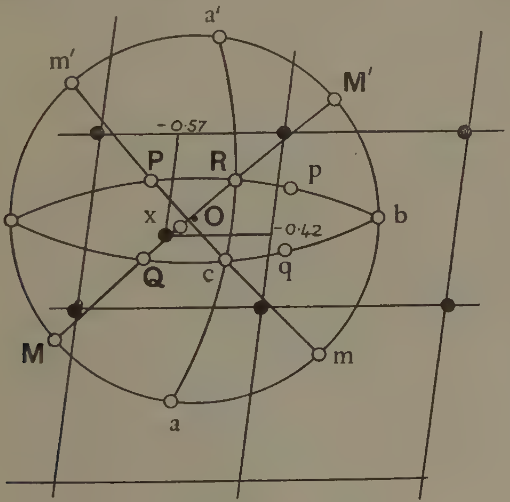

FIG. 77.

single crystal of the inosite derivative (as mentioned on p. 96) will serve as a basis for the discussion. The combined gnomono-stereogram of Fig. 77 shows that the face

lies in the zone MQR, but not at the intersection of this zone with the zone mcP. A glance at the projection is therefore sufficient to show: (1) that the indices of x are *not* (-1, -1, 2); and (2) that, whatever the indices are, they must be very complex—in view of the closeness of x to the true intersection point. *It is therefore clear that the gnomonic projection cannot be depended upon to yield the same indices as a process of logarithmic calculation.* It seems to the present writer an occasion on which an observer might well make a distinction in his treatment of a face like x of obviously complex indices and his usual treatment of a face with simple indices, if only for the reason that no finality is attached to the determination of indices which are not simple. Whether, for example, an observer is satisfied with the indices (-5, -7, 12) or prefers indices much more complicated (as being in better accord with the measured values) depends so much on subjective influences that the result, whatever it may be, can have very little objective value. Accordingly, would it not be better to determine the plane co-ordinates of the gnomonic pole, and express them as decimal fractions of the unit distances, *e.g.* x (-0.42, -0.57), rather than by indices? Such plane co-ordinates are, of course, not absolutely consistent with the tautozonality of x with MQR, but this is, perhaps, not a drawback, as tautozonality is of the nature of a fiction when examined from the narrowest possible point of detail.

It now remains to estimate the benefits which would follow any general discontinuance of the custom of computing theoretical angular values. Of the various types of calculation which constantly recur in crystallography, the development of a rectangular zone undoubtedly gives the least trouble, for it is generally a question of consulting the multiple tangent table. Of a somewhat higher and apparently equal order of difficulty are (1) the solution of a right-angled spherical triangle, and (2) the development of a non-rectangular zone (or sheaf of zones) by the cotangent form of the Millerian formula. The third and most time-consuming group of operations is well typified by the general solution of an oblique (plane or spherical) triangle. Now the general absence of the last type of calculation in systems down to the orthorhombic allows an inference that the amount of time saved by a refusal to compute theoretical angular values will not be very great in the five more symmetrical systems. In the monoclinic system, on the other hand, there will be a considerable saving, both because this system will presumably account for some 50 per cent. of all future crystals, and because the solution of an oblique plane triangle as well as several right-angled triangles will be obviated. It is of course in the anorthic system that the relief will be chiefly felt. Instead of computing a succession of oblique triangles (operations which have always to be carefully checked graphically in view of the especial liability to error in all complicated calculations), the single-circle worker can restrict his attention to the two or three triangles involved in the angular elements. The two-circle worker will be even more advantageously placed. By measuring the crystal from some face or other, which can be regarded as b(010)—it is presupposed he is carrying out original measurements and not repetition work—he will have three out of the five angular elements in his hand as measured data, and need only compute the two remaining angles (d and e). Further, if he wishes he can avoid calculations altogether; for by

adjusting his crystal anew by a face which he regards as $a(100)$, he can measure the two remaining angular constants. In the latter eventuality it is suggested that he should refrain from measuring the whole of the crystal afresh from $a(100)$, for it is difficult to see what good purpose could be served by any such multiplication of angles (interfacial or interzonal), even if they be direct observations as opposed to their theoretical derivatives.

Unfortunately no record was kept in the year 1906 of the time devoted to the calculations involved in the published description of the inosite compound; but an examination of the note-book is enough to indicate that the operation must have been protracted. Three four-pole-in-a-zone problems were solved by the 'theta'-formula and twenty-four oblique spherical triangles [something like 22 would have sufficed by another method]. If the indices had been determined by means of the gnomonic projection, and the logarithmic calculations confined to the computation of elements from five fundamental angles, the work would have been abbreviated to one four-pole-in-a-zone problem (by the cotangent formula) and the solution of three triangles: *i.e.* to one-sixth the original computations. If the elements had been improved by the use of the cotangent formula, the work rises to about one-fifth. But if the work involved in thinking out methods of computing a long series of spherical triangles be taken into account and set against the determination of indices by the gnomonic projection, then the fraction reverts to one-sixth. The case under review is perhaps not quite typical, as anorthic crystals are seldom measured quite so elaborately; but a comparison of other published accounts of anorthic crystals would seem to show that *a refusal to calculate what has been measured* would still lead to a saving of five-sixths of the time devoted to this, the only vexatious part of the investigation of an anorthic compound. And it will be realised that the whole geometrical investigation of an anorthic substance need only take up two, or three days at the outside—unless of course a large number of crystals are subjected to measurement.

10. **Crystal Drawings.**—In view of the great assistance unquestionably provided by a good drawing in any general comprehension of a crystal, it will be generally agreed that Haidinger's view—that the rapid acceptance of Haüy's contributions to crystallographic science was due more to the excellent drawings than to any other auxiliary device—expresses a general truth. Accordingly, apart from the obvious observation that a drawing may well be omitted in extremely simple cases of form development, it would only seem necessary to consider, and that quite briefly, the various alternative types of drawing. The type generally favoured at the present day is either the 'clinographic' drawing obtained from an axial cross (prepared according to rules described in most treatises) or the orthographic plan on an 'irrational' plane, which can be changed at will to suit the particular crystal (see Chapter V). But the fact must not be lost sight of that a less general and more easily prepared plan was much in vogue in the middle part of the last century. Brooke and Miller, *e.g.* only adopt a general position in the case of the cubic system; in all other systems the drawing is simply a plan—on the basal plane in the higher systems; on $b(010)$ in the monoclinic system and on the plane perpendicular to the vertical axis in the anorthic system. In other words, with the exception of the cubic system, and provided the monoclinic system be erected about the symmetry axis, the drawings are horizontal plans

in every case, *i.e.* the very figures that can be obtained with such a minimum of effort, either from the stereographic or from the gnomonic projection (an elegant plan of orthoclase, drawn from the gnomonic projection, is included in Neumann's *Beiträge zur Krystallonomie*). Now the only possible drawback to this method of depicting a crystal is that several zone edges may appear to be parallel, but as this is not likely to deceive an expert, the horizontal plan can scarcely be dismissed offhand in view of its simplicity of execution. Many mineralogists make a practice of giving a pair of drawings: (1) the horizontal plan, and (2) the general drawing obtained therefrom by the Goldschmidt method. This practice might well be imitated in the rare cases in which a laboratory product exhibits a highly complex form development, but as a rule either (1) or (2) should suffice. These alternatives are illustrated for the inosite compound in Figs. 80 and 81 at the end of this chapter.

11. **Cleavage, Specific Gravity and Topic Axes.**—The general desirability of determining the first two mentioned properties, whenever, possible, has long been recognised. The general practice of computing topic axes, on the other hand, has come in quite recently, for although according to Viola (*Trattato di Cristallografia*, p. 360) these constants were first suggested by Poletik in 1858, it is only as a result of Muthmann and Becke's independent ideas that they have been taken up by crystallographers. Tutton's well known researches on two large groups of isomorphous compounds (epitomised in his *Crystalline Structure and Chemical Composition*, 1910), and the author's investigations on crystal structure (*J. Chem. Soc.*, 1906, **89**, 1120; *Min. Mag.*, 1907, **14**, 235; 1908, **15**, 42; *Zeitsch. Kryst. Min.*, 1908, **45**, 1) are typical illustrations of the usefulness of these constants in any comparison of the members of an isomorphous series. But the extension of the principle to substances exhibiting a more remote resemblance is, perhaps, unjustifiable in view of the fact that so little is known concerning the fine details of the average crystal, until it has been subjected to the test of the X-ray method. Theoretical angles and theoretical topic-axes would appear to belong to the same category; both have played a useful rôle in the past (the latter up to quite recently), and both may re-acquire a measure of interest in the future. These possible future requirements would appear to be sufficiently well provided for, if measured angles, cleavage and specific gravity (the latter being so important from the X-ray standpoint) be put on record.

12. **Needles, Scales and Other Complexities.**—Many substances, unfortunately, defy an exact geometrical description for reasons which are now sufficiently well understood. According to the Bravais prin-

ciple[24]—a principle which has only recently found general acceptance as a result of the convincing studies of G. Friedel (*Bull. Soc. franç. Min.*, 1907, **30**, 326)—two extreme varieties of structure are necessarily responsible for an acicular or scaly habit respectively, the result being a single dominating zone devoid of terminal plane facets in the one case, and a single parallel pair of faces devoid of peripheral plane facets in the other. Future developments may make it possible to measure by means of the *X*-ray spectrometer what cannot be investigated by the goniometer. In the meantime it seems desirable to characterise such compounds as far as possible by optical methods. The measurement to three or four units in the third decimal place of the principal refractive indices by a microscope-method, which has been recently perfected by F. E. Wright, would seem to be desirable. Further, as a platy or scaly habit makes a substance particularly suitable for examination by means of the Fedorov 'Universal Stage' (*cf.* V. V. Nikitin, '*La méthode universelle de Fedoroff*' [Russian, 1911: French translation by L. Duparc and V. de Dervies, 1914]; F. E. Wright's '*Methods of petrographic-microscopic research*,' 1911), and as the optical data so obtained are classifiable, there is an obvious possibility here of extending the range of substances which can be absolutely identified by means of a characteristic crystalline property. With regard to the term 'other complexities,' it need only be mentioned that

[24] According to the Bravais principle (or law) the decisive formative influence in crystallography is a preference for the most densely packed structure-planes as boundary surfaces on a growing crystal. Its general correctness has not been invalidated by *X*-ray work (this work has not really advanced far enough to 'prove' it efficiently). The varied fortunes of the principle are only equalled by those of the gnomonic projection. First adumbrated by Frankenheim, the principle was enunciated and worked out in all its consequences by Bravais. Its value was, however, not appreciated, and it would seem to have been ignored altogether outside France. An imperfect form of the idea was conceived by Junghann; developed as the 'Law of Complication' by Goldschmidt; and finally generalised by Fedorov. It was then pointed out by Friedel ('Etude sur les Groupements Cristallins,' *Bull. Soc. de l'Industrie minérale*, 1904, **3—4**, pp. 486, especially p. 131) that the idea of complication only takes cognisance of the 'simple multiple' part of the law of simple multiple intercepts, and not of the 'intercepts' themselves, and is to that extent incomplete. The Bravais principle on the other hand includes both. The law of complication really implies that the faces developed on a crystal are those which have the simplest indices. The Bravais principle, on the other hand, whilst demanding such faces as a rule, contains the further implication that a gross inequality of the three structural dimensions will disturb this primitive numerical simplicity and bring about the development of certain planes relatively more thickly beset with particles than those of simpler indices. The profound studies of H. Baumhauer ('Die neuere Entwickelung der Krystallographie,' 1905, 137 sqq.) and of Friedel bear out the requirements of the Bravais principle, and support Friedel's conclusion that it presents the right correlation of form and structure, in so far as the Bravais lattice, as opposed to the Sohncke-Fedorov-Schönflies-Barlow 'point-system,' represents the ultimate scheme of architecture. It has thus become clear that an enunciation of the fundamental morphological law as 'The Law of Rational Intercepts' only covers about half the truth; for it is only an alternative way of stating that a crystal never exhibits symmetry axes, save those of two-, three-, four- or six-fold symmetry, and does not include what is at least equally remarkable: that a structure, subordinate to these limitations of symmetry, proceeds to limit its boundary planes during growth to those that are relatively thickly beset with particles. It is really due to this second limitation that the first was discovered, and that crystallography to-day presents such a body of co-ordinated facts, as can perhaps only be challenged by the chemistry of the carbon compounds.

any occasional features which are likely to afford special trouble to a future investigator (any intricate distortion of crystal habit, twinning, and so on) should be adequately described.

13. **Optics and Structure.**—A detailed consideration of these two important branches of crystallographic investigation must be omitted as lying outside the scope of this work.

14. **A Typical Investigation.**—It may now be useful to outline the course of a typical investigation according to the principles here advocated. A crystal of the supposedly new inosite compound is sketched and measured on the single-circle goniometer (if a two-circle instrument is not available) up to the point at which the faces can be plotted on a stereographic projection. In the present case four or five zones are sufficient. A projection (Fig. 78) is then prepared.

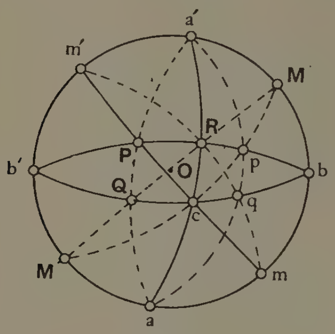

FIG. 78.

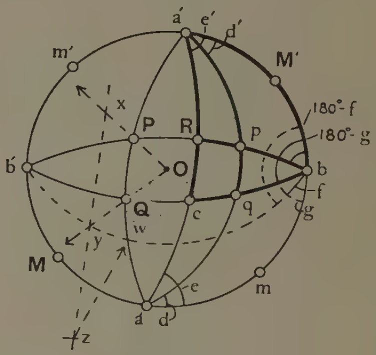

FIG. 79.

The crystal appears to be anorthic and so does the projection, but both may be deceptive. The part C of the protractor is now applied to the analysis of such supplementary zones as those represented by interrupted lines in the diagram. The new interfacial angles are read off by means of the small circles of the protractor and compared with each other and with those measured on the goniometer. Although there is an occasional correspondence here and there the angular values as a whole cannot be reconciled with any higher system of crystallisation. The system is now confirmed by a simple optical examination; one or more extra crystals are measured; the means are extracted; a consistent set of indices is determined (by means of the gnomonic projection if necessary); a drawing is prepared from the gnomonic or stereographic projection; and there only remains the calculation of elements (in this case 'angular' since the crystal is anorthic). The preliminary adjustment of fundamental angles by means of the cotangent formula is, perhaps, unnecessary in a case like this

in which the measurements do not show great fluctuations. The three sides of the triangle $a'Rb$ (*cf.* Fig. 79) may, for example, be selected at the outset, and the angles $Ra'b$ ($= e' = e$) and $a'bR$ are computed. The angle cR is now invoked as a fourth fundamental, and the triangle $a'cb$ solved for the angle cba' ($= 180° - g$). The interzonal angles at $b(100)$ are how developed by the Millerian cotangent formula, the object being to derive the value $(180° - f)$ as indicated in the Figure. The angle $a'q$ (or, alternatively, the angle bq) is now added as a fifth fundamental, and the triangle $a'qb$ solved for the angle $qa'b$ ($\doteq d' = d$). The angular elements d, e, ab, f, g are complete for $ab = 180° - a'b$. *They must now be checked graphically.* This operation, in so far as it relates to the angles d, e, ac and g, needs no comment, but the angle f involves a zone (drawn as an interrupted arc in Fig. 79) which does not occur on the crystal and has therefore to be deduced graphically from the other zones before the check can be applied. There are the usual two ways: (1) a *formal* recourse to the gnomonic projection (the value so found was $f = 34\frac{1}{2}°$ instead of $34° 37'$ by calculation—the agreement is unusually good); (2) an *informal* recourse to the gnomonic projection as indicated on the left side of the Figure. A convenient line xyz is drawn parallel to the diameter aOa'. Rays Ox and Oy are drawn through the stereographic poles of $P(\bar{1}\bar{1}1)$ and $Q(0\bar{1}1)$. The length xy is carefully repeated as a length yz. The ray zO is drawn so as to intersect the zone $aQPa'$ in the potential face pole $w(1\bar{1}1)$. This point determines the lie of the zone $(010)-(111)-(101)-(1\bar{1}1)-(0\bar{1}0)$, *i.e.* the interrupted arc of Fig. 79 (which need not be drawn); and its inclination to the vertical, *i.e.* the angle f can be read through the part C of the protractor. The value so found (33°) is not particularly good, but it is sufficiently close to the computed value $34° 37'$ as a check. NOTE.—The line xyz is of course an informal zone line PQ (*i.e.* one with an unspecified radius of the circle of projection) as opposed to *the* gnomonic zone line (*i.e.* one belonging to a projection with the standard 5 cms. radius).

15. **A Typical Description.**—Any eventual adoption of the preceding body of suggestions will naturally lead to a considerable saving of the researcher's time (a point which will be considered below). It will also allow him to express his results more economically. The omission of details referring to the number of angles measured, limits and theoretical values, disposes of about three-quarters of the 'table of angles,' and incidentally suggests a linear in place of the 'columnar' arrangement. Then, an omission of superfluous angles (except in the cases where, say, an extra zone might be held to be useful), reduces what is left by about one-half. [This great decrease would be scarcely altered by the inclusion of computed angles, provided such values are expressed as the difference in minutes (with minus sign if necessary)

between the computed and the observed values. Thus 36°43′(5) might signify that the computed value is 36°48′, whilst 36°43′(−5) might imply a theoretical value 36°38′.] Following is a representative description, in which it is assumed that a researcher has adopted some recognised scheme of lettering, but believes it to be advisable to compute theoretical angular values. The various zones are printed as separate paragraphs, but their simple separation by semi-colons, or better still by strokes [thus, / . . /], would probably have served equally well.

DIBROMOINOSITE TETRA-ACETATE. $C_6H_6Br_2(O.CO.CH_3)_4$.—This substance crystallises in the anorthic system, the elements being, d = 35° 37′, e = 67° 50′, ab = 97° 31′, f = 34° 37′, g = 63° 53′, and

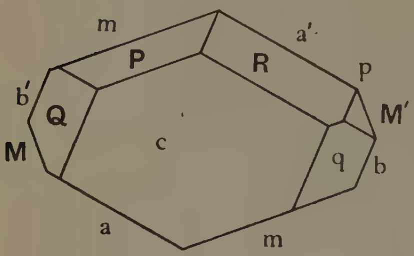

FIG. 80.

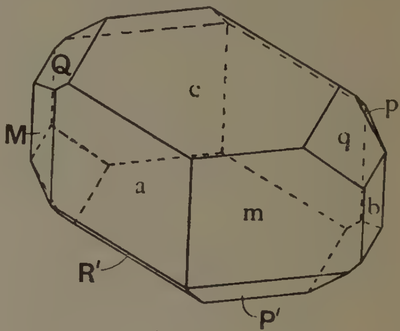

FIG. 81.

the forms $b, m, a, M, q, c, Q, R, P,$ and rarely p (as illustrated by Fig. 80 [or alternatively by Fig. 81]). Following are the mean results of measurement of [say three] twelve crystals:

bm = 47° 49′(−5), ba = *97° 31′, bM = *139° 33′;
bq = 39° 38′ (1), bc = 72° 47′ (−4), bQ = 120° 10′ (7);
bp = 36° 49′ (6), bR = 66° 40′ (9), bP = 115° 12′ (11);
mc = *59° 17′, mP = 124° 45′ (6);
ac = *67° 48′, aR = *115° 57′.

Cleavage, a (perf.); D(18°/4°) = 1·713. [Optics, as required.]

The above description is seen to take up twelve lines of print, as opposed to some 64 lines in the original. The addition of four lines for optics brings the two totals to sixteen and 68 respectively. Including the illustrative figure the space required is about one-third of that of the original. The question of the relative areas of the two descriptions, though not negligible, is, however, of subordinate importance. The really vital question is whether the new description (with or without the computed angular values) is adequate for all reasonable purposes. This question must be left to a reader to decide.

16. **Note on the Presentation of Two-Circle Measurements.**—The only difference lies in the table of angles, which has now to be arranged on a three-line system. The following values refer to the monoclinic *caes*ium magnesium chromate, $Cs_2Mg(CrO_4)_2 6H_2O$ (*J. Chem. Soc.*, 1911, **99**, 1328):—

	$b(010)$	$m(110)$	$o(111)$	$q(011)$	$c(001)$	$p(\bar{1}11)$	$s(\bar{2}01)$
ϕ .	0°0′	*54°31′	63°10′(12)	30°33(3)	90°1′(·1)	320°46′(10)	270°0′
ρ .	90°0′	90°0′	47°17′(11)	*29°35′	*16°7′	32°24′(·13)	47°28′(·13)

17. **General Suggestions.**—In view of the somewhat lengthy nature of this general examination of crystallographic routine, it seems desirable to summarise the more general suggestions. These are:

(1) That crystals be measured with a minimum of elaboration.

(2) That an accurate stereographic projection be prepared as soon as the measurements allow, and the system determined by a graphical comparison of angles in projection and confirmed at an early stage by a simple optical examination.

(3) That two (or at the most three) crystals be measured, each crystal being treated in the same way (not some zones on one, and other zones on the others).

(4) That the indices be determined by a time-saving method of precision (in general the gnomonic projection, or its stereographic equivalent in the simpler cases; and by the multiple tangent table in the particular case of rectangularity); that the exceptional case (complexity of indices)—amounting, perhaps, to one case in a thousand faces—be treated in an exceptional way (by the sine form of the Millerian formula, or its rectangular equivalent—the supplementary multiple tangent table of p. 149, or by a decimal citation of the gnomonic plane co-ordinates).

(5) That a useful set of elements be calculated with the utmost care from fundamental angles, which have been previously mutually adjusted as far as circumstances allow.

(6) That the mean observed angles be published without any citation of limits; and:

(7) That the practice of computing theoretical angular values (apart from those involved in the elements) be discontinued.

The immediate result of any substantial adoption of the above suggestions would be a considerable saving of time. In the case of single-circle goniometry, careful estimates allow an expectation that the average crystallographic investigation need not occupy one-half the time demanded by present-day customs.[25] But this is perhaps not

[25] The net effect in two-circle practice is more difficult to estimate on account of the varied degrees of elaboration adopted by different workers. The adoption of exact graphical methods does not come into any estimate since they are in general use amongst two-circle workers. On the other hand the time saved by omitting to

the more important result which should follow any suitable adjustment of means to ends, for the discontinuance of such tedious operations as the computation of long lists of angles on the one hand, and the adoption of a more varied equipment on the other, can scarcely fail to lend an enhanced interest to the practical study of a crystal.[26]

EXERCISES.

**Exercise* 68.—Given an anorthic crystal, with the elements $a : b : c = 1{\cdot}0644 : 1 : 0.9153$, $\alpha = 112°10'$, $\beta = 116°7'$, $\gamma = 74°3'$ and the forms $a(100)$, $b(010)$, $c(001)$, $m(110)$, and $q(011)$. Derive the formulae for the conversion of this type of elements into the Fedorov angular form, or, if preferred, into the interfacial angles from which they were originally computed.

Exercise 69.—Derive appropriate Fedorov angular elements for a monoclinic crystal and discuss the question of their general adoption.

Exercise 70.—In the topaz crystal of Figs. 58—59 (p. 70) the mean measured angles in two important zones are: $cf(001 : 011) = 43° 52'$, $cy(001 : 021) = 62° 25'$, $ci(001 : 113) = 34° 11'$, $cu(001 : 112 = 45° 35'$, $co(001 : 111) = 64° 2'$. Compute accurate elements.

Exercise 71.—Combine with the foregoing the angles: $br(010 : 130) = 32° 26'$, $bl(010 : 120) = 43° 20'$, $bm(010 : 110) = 62° 8'$, and deduce a still more accurate set of elements.

Exercise 72.—Given the zone $m(110)$, $o(111)$, $c(001)$, $p(\bar{1}\bar{1}1)$ in a monoclinic crystal and the measured angles $mo = 33° 37'$, $mc = 87° 17'$, $mp = 144° 33'$. Effect a reduction to the two fundamentals mo and mc.

**Exercise* 73.—Given in the same crystal $mocx$ $(\bar{1}\bar{1}2)$ with $mx = 124° 12'$. Effect a similar reduction from $mocx$, and finally adjust the two reductions $mocp$ and $mocx$.

compute theoretical angles will in every case be as much as in single-circle goniometry, and in some cases much more. If these latter cases be left out of account the average saving of time can scarcely exceed 30 per cent. and may be somewhat less.

[26] In conclusion attention may be drawn to the economic benefits which must flow from any general adoption of a more restrained method of presenting crystallographic constants. It is, perhaps, not generally realised that the type-setting of tabular matter involves a considerable strain on the part of the compositor, resulting in numerous errors unless he works at a slower rate, and that the cost per page may be as much as three times the normal. At the present rate of printing, the difference between the old and the new table of angles for the inosite compound is of the order of £2—a not inconsiderable figure, especially when it may have to be multiplied twenty or a hundred times yearly in the case of a Journal partly or wholly devoted to matters of crystallographic interest. This financial side of the question will, however, become a matter of supreme importance when the time comes to issue the next work of reference on crystal constants. The number of compounds may presumably have grown from 10 to 15 thousand. Will this increase of the material necessarily imply a corresponding expansion either in the number or size of the component volumes? The present writer was requested some years ago to outline a detailed scheme for such a work and came to the conclusion that the only circumstance that could stand in the way of a one-volume work, embracing present-day materials, is the printing of long lists of theoretical angular values; that a two-volume work would be ample to contain some 15,000 entries; and that, if the fullest use be made of the previous labours of Rammelsberg, Groth and Fedorov, a small group of collaborators could produce a handy two-volume work of reference (say, 'mineral' and 'organic'), previously unexcelled in essential comprehensiveness, at a very moderate cost.

CHAPTER IX.

ANSWERS TO EXERCISES.

Note.—In most cases the exact values are given to the nearest minute. A graphical solution may be considered satisfactory in most cases if it does not differ from the exact value by more than $\frac{1}{2}°$. In an exceedingly rare case the permissible error (*i.e.* an error which does not necessarily indicate careless work) may be $3/4°$ or even $1°$. Values derived from the multiple tangent table should not differ from the answers given below by more than $1'$ (or, occasionally, $2'$ if a double consultation has been involved).

Exercise 4.—The indices are: $x = (241)$, $y = (243)$.

Exercise 5.—The rho-values are: $39° 57'$ (113); $59° 10'$ (223); $68° 18'$ (111); $56° 27'$ (335) and $68° 18'$ (256).

Exercise 6.—The indices are: $l(113)$, $u(112)$, $y(021)$, $v(134)$, $w(243)$.

Exercise 7.—The dimensions of the rectangular gnomonogram are about $5·7 \times 7·7$ cms., with the corner $c(001)$ about 1·3 cms. in front of the centre of the projection. Following are the central angular distances or rho-values: $(111) = 65° 46'$, $(\bar{1}01) = 47° 30'$, $(013) = 29° 50'$, $(112) = 56° 20'$. The angle $(001):(121) = 58° 33'$.

Exercise 8.—Angle $(001):(121) = 42° 38'$.

Exercise 10.—The stereographic radius cm already passes through the origin of co-ordinates (the gnomonic pole of c) and therefore intercepts the zone-line $(011):(111)$ in the gnomonic pole of (111) and thus defines one of the sides of the primitive gnomonogram.

Exercise 11.—Here again there is no difficulty, but the radius Om has now to be translated till it passes through the gnomonic pole of $c(001)$; it then intercepts the horizontal line, drawn through the gnomonic pole (101), in the gnomonic pole (111). The three gnomonogram corners (001) — (101) — (111) are thereby known and the problem is virtually solved.

Exercise 12.—The five faces have, respectively, the indices, (130), (120), $(1\bar{1}0)$, $(1\bar{2}0)$ and $(1\bar{3}0)$.

Exercise 13.—The three faces have, respectively, the indices (100), $(2\bar{3}0)$ and $(1\bar{2}0)$.

Exercise 14.—The indices are: (100), $(2\bar{1}0)$, $(1\bar{1}0)$ and $(1\bar{2}0)$.

Exercise 15.—The theoretical angular values are: $bn = 25° 16'$, $bL = 120° 8'$, $bG = 127° 15'$, $bM = 138° 15'$.

Exercise 16.—The indices are $p(012)$ and $Q(0\bar{1}1)$.

Exercise 17. In this particular zone of a monoclinic crystal *passing through the centre of the projection,* the gnomonic method is used, since the gnomonic poles can be plotted directly from the data. [There is no need to make the primitive gnomonogram.] The indices are $l(201)$, $R(\bar{1}01)$, $s(\bar{2}01)$ and $t(\bar{3}01)$.

Exercise 18.—The theoretical value of aR is $105° \; 27'$.

Exercise 20.—In the general figure the vertical dimension of the face $a(100)$ is determinable from the consideration that its two face-diagonals are respectively parallel to the edges [100 : 011] and [100 : 0$\bar{1}$1]. Either of course is sufficient. As a matter of principle that should be chosen which intercepts the vertical edge at the greater angle. The actual choice depends on the position from which the cube is drawn.

Exercise 23.—The poles of the five prism faces have, of course, to be plotted at angles of 72° on the primitive from which the regular pentagonal plan is drawn in the usual way. The final figure is obtained on the usual mechanical lines.

Exercise 30.—The guide-line is seen to pass very close by the gnomonic pole (0$\bar{1}$1), and this face would become excessively foreshortened if drawn from the standard position. The needed improvement can be effected by increasing the angle of tilt from $9\frac{1}{2}°$ to 13° or 15°; or, alternatively, by adopting a rotation of 10° (instead of 20°), since the face b(010) is not developed and therefore cannot be foreshortened thereby; or, finally, by a suitable recourse to both types of adjustment.

Exercise 32.—$AB = 29° \; 29\frac{1}{2}'$.

Exercise 33.—$AD = 117° \; 33'$.

Exercise 34.—$AD = 130° \; 53\frac{1}{2}'$.

Exercise 35.—$AD = 106° \; 24'$.

Exercise 36.—Phi-value, $C(21° \; 44'$.

Exercise 37.—Phi-value, $C \; (61° \; 26')$.

Exercise 38.—Phi-value, $D \; (136° \; 27')$.

Exercise 39.—The value 58° 4′ has first to be subtracted from the standard phi-values of B and C. The cotangent formula then leads to a correspondingly modified phi-value $D \; (164° \; 27\frac{1}{2}')$. The standard phi-value is, therefore, $164° \; 27\frac{1}{2}'$ plus $58° \; 4'$, *i.e.* $222° \; 31\frac{1}{2}'$.

Exercise 40.—$AD = 117° \; 33'$ (following the special formula, $\cot AB + \cot AD = 2 \cot AC$, obeyed by Exercises 40—45).

Exercise 41.—$AD = 110° \; 36'$.

Exercise 42.—$AB = 39° \; 54\frac{1}{2}'$.

Exercise 43.—$AB = 39° \; 54\frac{1}{2}'$.

Exercise 44.—$AB = 28° \; 18\frac{1}{2}'$.

Exercise 45.—$AB = 40° \; 3\frac{1}{2}'$.

Exercise 46.—The following harmonic quartettes permit the evaluations required: A, B, C, F (whence $AF = 110° 33'$); A, B, D, C (whence $AD = 64° 48\frac{1}{2}'$); A, D, C, E (whence $AE = 95° 19'$); A, C, F, G (whence $AG = 133° 11'$).

Exercise 47.—Phi-value of $C = 21° 44'$.

Exercise 48.—Phi-value of $D = 106° 7'$.

Exercise 49.—Phi-value of $D = 154° 34'$.

Exercise 50.—The harmonic quartettes are: A, B, C, E (whence phi-$E = 138° 48\frac{1}{2}'$); A, D, B, C (whence phi-$D = 33° 6\frac{1}{2}'$); A, D, C, F (whence phi-$F = 153° 49\frac{1}{2}'$); A, E, F, G (whence phi-$G = 161° 8'$).

Exercise 51.—The standard phi-value of A has first to be subtracted from those of B and C. The amended phi-values (*i.e.* those reckoned from the pole A as zero-phi) then follow simply from the harmonic quartettes A, B, C, D and A, B, E, C; and the final addition of phi-A leads to the standard phi-values, $D = 206° 2'$, $E = 59° 13\frac{1}{2}'$.

Exercise 52.—This is a typical calculation involved in the calculation of elements or in the solution of an oblique spherical triangle. As explained in the chapter the four zone-symbols (whatever they may be) can be replaced (for purposes of calculation) by the four face indices, $a(010)$, $b(011)$, $c(001)$, $d(0\bar{1}1)$. And as the four face-indices are evidently harmonic, the four zones must also be harmonic. Therefore, $\cot AB + \cot AD = 2 \cot AC$. Evaluation shows that $AC = 67° 50'$.

Exercise 53.—Harmonic; $AC = 97° 58'$.

Exercise 54.—Harmonic; $AC = 105° 57'$.

Exercise 55.—As this is a case which, on inspection, does not seem to be harmonic, the numerical co-efficient p/q of the general cotangent formula has to be worked out from the given indices. As $p/q = 1/4$, we have $\cot AB - 4 \cot AC = -3 \cot AD$; whence $AD = 98° 53'$.

Exercise 56.—The first four values can be read off in a sequence, namely, $45° 0' \rightarrow 63° 26' \rightarrow 71° 34' \rightarrow 75° 58' \rightarrow 78° 41'$. For the next four values we have: $26° 34'$ (col. 1) $\leftarrow 45° 0'$ (col. 2); $18° 26'$ (col. 1) $\leftarrow 45° 0'$ (col. 3); $14° 2'$ (col. 1) $\leftarrow 45° 0'$ (col. 4); $11° 19'$ (col. 1) $\leftarrow 45° 0'$ (col. 5). It is of the utmost importance to grasp the significance of the above lesson, namely, that a sequence of simple faces on that side of the unit face (in this case 110) which is remote from the basal face (in this case 100) can be read as a single line; whilst faces adjacent to the basal face involve explorations in different parts of the table. This slight inconvenience could be avoided in practice by adopting a face $b(010)$ as basal for the faces lying between $m(110)$ and $a(100)$, the complementary angles being thereby obtained; but this inversion is not worth while in actual crystallographic work as long sequences of faces such as (510), (410), (310), (210), (110) are never (or, at any rate, only most rarely) observed.

Exercise 57.—70° 31′, 76° 44′, 35° 16′, 25° 14′.

Exercise 58.—23° 39′, 33° 18′, 69° 10′, 75° 45′; with regard to the face (023), the indices show: (1) that the second and third columns are alone concerned, and (2) that the angle required must be less than the unit angle 52° 43′ (the third index being greater than the second). We have, then, to look up 52° 43′ in the third column and read the answer 41° 13′ from the second column.

Exercise 59.—Similar to the last item of the previous exercise. The values are 32° 48′, 47° 3′, 52° 12′, 67° 31′.

Exercise 60.—With the exception of the last-mentioned face, each evaluation involves a double consultation of the table. Thus (109) may be regarded as (103) × (103); (108) = (104) × (102); (106) = (103) × (102); (308) = (304) × (102) or (302) × (104); (508) = (504) × (102) or (502) × (104). The values so determnied are 10° 9′, 11° 23′, 15° 2′, 31° 8′, 45° 11′. The face (23 . 0 . 24) lies outside the multiple tangent table, and the angle has to be computed logarithmically, but its citation in the original literature can, perhaps, not be regarded as a serious contribution to crystallography.

Exercise 61.—The zone being rectangular, the values are readily found to be 72° 24′, 38° 14′, 27° 43′; 67° 4′, 46° 25′.

Exercise 62.—The zone being rectangular, the values are 27°45′, 38° 16′, 46° 27′, 52° 45′, 57° 38′, 21° 31′, 49° 47′ (or possibly 49° 48′).

Exercise 63.—The zone being rectangular, the values are 35°36′ and 25°31′.

Exercise 64.—(1) Since $p/q = \frac{1}{2}$ and 2 respectively, we have $rx = 17°58'$ and $ry = 52°21'$; (2) $p/q = \frac{1}{2}$, therefore $sv = 22°1'$; (3) $p/q = 2$, therefore $cw = 63°49'$; (4) $p/q = 2$, therefore $ug = 42°57'$.

Exercise 65.—The angles, *rx*, *ry*, and *sv* [or equally well, their phi-values] because the zones concerned remain rectangular.

Exercise 66.—In any case in which it is necessary to refer to a gnomonic projection for the correct p/q coefficients, it is better to reckon the phi-values from the pole lying in the plane of symmetry (in this case 302) rather than from *b*(010)—for the reason that the pole *b*(010) lies at infinity. The initial phi-value thus acquires the complementary value 22°53′, and a glance at a gnomonic projection shows that the p/q values are $\frac{1}{2}$, 2 and 3 respectively. The phi-values as reckoned from (302) are accordingly 11°55′, 40°10′ and 51°41′. The standard phi-values (complementary to these) are accordingly 78°5′, 49°50′ and 38°19′.

Exercise 67.—Here again it is expedient to adopt the face 90° distant from *b*(010), *i.e.* in this case the face ($\bar{1}01$) as a phi-basis. This is effected by subtracting 270° from the given value 302°24′. The gnomonic projection shows that the numerical coefficients are $\frac{1}{2}$ in the case

of $v(\bar{2}12)$ and 3 in the case of $z(\bar{1}31)$. We have then: 17°36′ (first col.) ← 32°24′ (second col.); and 32°24′ (first col.) → 62°17′ (third col.). Adding to each the 270° initially subtracted, we finally obtain 287°36′ and 332°17′ as the standard phi-values.

Exercise 68.—The computation of the interfacial angle *ab* (or *bc* or *ca*), follows readily by the solution of the oblique spherical triangle *abc*, defined by the supplements of the axial angular values, α, β, γ. It is at this stage that the real difficulty is encountered. It can only be met by the computation of the interzonal angles at *a* and *b* (*cf.* Fig. 94) by the following formulae:

$$\tan d = \frac{\sin(180^\circ - \alpha)}{c/b + \cos(180^\circ - \alpha)}; \quad \tan f = \frac{\sin(180^\circ - \beta)}{c/a + \cos(180^\circ - \beta)},$$

with due regard to the fact that the cosine of any angle $x = -\cos(180^\circ - x)$. The values *e* and *g* are directly given by $e = (180^\circ - \alpha)$ and $g = (180^\circ - \beta)$ respectively, but in order to avoid the possibility of errors in the evaluation of the above formulae, it is always advisable as a check to compute the component angles, *e-d* and *g-f*, independently by the formulae:

$$\tan(e-d) = \frac{(c/b)\sin(180^\circ - \alpha)}{1 + (c/b)\cos(180^\circ - \alpha)}; \ \tan(g-f) = \frac{(c/a)\sin(180^\circ - \beta)}{1 + (c/a)\cos(180^\circ - \beta)}.$$

This derivation of the Fedorov angular elements completes the difficult part of the work, other interfacial or interzonal angles being easily computed therefrom.

NOTE.—The above formulae can be re-arranged into:—

$$\cot d = (c/\sin\alpha) - \cot\alpha; \ \cot f = (c/a \sin\beta) - \cot\beta;$$
$$\cot(e-d) = (1/c\sin\alpha) - \cot\alpha; \ \cot(g-f) = (a/c\sin\beta) - \cot\beta.$$

Exercise 69.—The angles *e* and *ab*, being in this case 90° by symmetry, are now discarded so that the Fedorov angular elements take the form *d, f, g* where (by symmetry) *d* is the angle whose cotangent is equal to the axial ratio $c:b$, *f* is equal to the interfacial angle 100 : 101, and $g = 100:001 = (180^\circ - \beta)$. With regard to the question of their adoption: They are clearly more suitable than the axial-ratio form, since no computations are involved in the preparation of an accurate gnomonic or stereographic projection, and they could with great advantage be adopted in any comprehensive work of reference. On the other hand, as hinted in the text, there are not the same unanswerable demands for a change as in the anorthic system. Further, their computation by the author of an original paper would frequently involve the solution of an oblique triangle, not always demanded by the orthodox axial-ratio form of elements. It is on such practical grounds as these that the present writer came to the conclusion that

the axial-ratio form might well remain. The monoclinic system holds the central position in any discussion of elements. On one side (the anorthic system) there would seem to be insistent demands for a change. On the other side (the orthorhombic system) there would seem to be cogent reasons against the adoption of Fedorov angular elements (having the form d, f, since both e and g are now 90°). In the orthorhombic case a more appropriate change would be to adopt the angles given in Dana's *System*, immediately after his axial ratios. And so on, up to the cubic system. In conclusion the opinion may well be expressed that the use of the element α in the rhombohedral system should be discontinued in favour of the angle cr (111 : 100), since an accurate projection cannot be made without a preliminary back calculation (in this case not very serious) to some such angle as cr.

Exercise 70.—The adjusted value of the fundamental, as given in the text on a previous page, is 63°54′. The adjusted value of cf is 43°48′ (neglecting a half minute). From these two values axial ratios may be computed.

Exercise 71.—The adjusted value of bm is 62°11′. We now have three fundamentals cf, co and ocf (= bm), and the problem is to adjust one of these (preferably co—why?) in terms of the other two. We solve the right-angled triangle *twice* as follows: (1) taking co and cf, we obtain ocf = 61°59′; (2) taking co and ocf (*i.e.* measured bm), we obtain cf = 43°36′. The mean ocf (measured and computed) is 62°5′ (its cotangent = 0·5298 = $a : b$). The mean cf (measured and computed) is 43°42′ (its tangent = 0·9556 = $c : b$).

Exercise 72.—The four poles are harmonic, therefore cot mo + cot mp = 2 cot mc. Putting in the measured values mo and mp we compute mc (87°9′), then putting in measured values mp and mc we compute mo (33°42′). Taking the means of mo and mc, measured and computed, we get finally mo = 33°39½′, mc = 87°13′.

Exercise 73.—Similar to preceding but as $m\ o\ c\ x$ do not appear to be harmonic, the coefficient p/q (actually 1/3) has to be evaluated by cross-multiplication; whence cot mo —3 cot mc = —2 cot mx, and cm (computed) = 87°14′, and mo (computed) = 33°40′. Accordingly, the means are mo = 33°38½′ . mc = 87°15½′. The final reduction is effected by taking the means of the original, measured values of mo and mc, the computed values from p and the computed values from x. Thus

$$mo = 1/3(33°37' + 33°42' + 33°40') = 33°40';$$
$$mc = 1/3(87°17' + 87°9' + 87°14') = 87°13'.$$

TABLE OF NATURAL COTANGENTS (AND TANGENTS).

NATURAL COTANGENTS.

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	′
0	∞	57·2900	28·6363	19·0811	14·3007	11·4301	9·51436	8·14435	7·11537	60
1	3437·75	56·3506	28·3994	18·9755	14·2411	11·3919	9·48781	8·12481	7·10038	59
2	1718·87	55·4415	28·1664	18·8711	14·1821	11·3540	9·46141	8·10536	7·08546	58
3	1145·92	54·5613	27·9372	18·7678	14·1235	11·3163	9·43515	8·08600	7·07059	57
4	859·436	53·7086	27·7117	18·6656	14·0655	11·2789	9·40904	8·06674	7·05579	56
5	687·549	52·8821	27·4899	18·5645	14·0079	11·2417	9·38307	8·04756	7·04105	55
6	572·957	52·0807	27·2715	18·4645	13·9507	11·2048	9·35724	8·02848	7·02637	54
7	491·106	51·3032	27·0566	18·3655	13·8940	11·1681	9·33155	8·00948	7·01174	53
8	429·718	50·5485	26·8450	18·2677	13·8378	11·1316	9·30599	7·99058	6·99718	52
9	381·971	49·8157	26·6367	18·1708	13·7821	11·0954	9·28058	7·97176	6·98268	51
10	343·774	49·1039	26·4316	18·0750	13·7267	11·0594	9·25530	7·95302	6·96823	50
11	312·521	48·4121	26·2296	17·9802	13·6719	11·0237	9·23016	7·93438	6·95385	49
12	286·478	47·7395	26·0307	17·8863	13·6174	10·9881	9·20516	7·91582	6·93952	48
13	264·441	47·0853	25·8348	17·7934	13·5634	10·9528	9·18028	7·89734	6·92525	47
14	245·552	46·4489	25·6418	17·7015	13·5098	10·9178	9·15554	7·87895	6·91104	46
15	229·182	45·8294	25·4517	17·6106	13·4566	10·8829	9·13093	7·86064	6·89688	45
16	214·858	45·2261	25·2644	17·5205	13·4039	10·8483	9·10646	7·84242	6·88278	44
17	202·219	44·6386	25·0798	17·4314	13·3515	10·8139	9·08211	7·82428	6·86874	43
18	190·984	44·0661	24·8978	17·3432	13·2996	10·7797	9·05789	7·80622	6·85475	42
19	180·932	43·5081	24·7185	17·2558	13·2480	10·7457	9·03379	7·78825	6·84082	41
20	171·885	42·9641	24·5418	17·1693	13·1969	10·7119	9·00983	7·77035	6·82694	40
21	163·700	42·4335	24·3675	17·0837	13·1461	10·6783	8·98598	7·75254	6·81312	39
22	156·259	41·9158	24·1957	16·9990	13·0958	10·6450	8·96227	7·73480	6·79936	38
23	149·465	41·4106	24·0263	16·9150	13·0458	10·6118	8·93867	7·71715	6·78564	37
24	143·237	40·9174	23·8593	16·8319	12·9962	10·5789	8·91520	7·69957	6·77199	36
25	137·507	40·4358	23·6945	16·7496	12·9469	10·5462	8·89185	7·68208	6·75838	35
26	132·219	39·9655	23·5321	16·6681	12·8981	10·5136	8·86862	7·66466	6·74483	34
27	127·321	39·5059	23·3718	16·5874	12·8496	10·4813	8·84551	7·64732	6·73133	33
28	122·774	39·0568	23·2137	16·5075	12·8014	10·4491	8·82252	7·63005	6·71789	32
29	118·540	38·6177	23·0577	16·4283	12·7536	10·4172	8·79964	7·61287	6·70450	31
30	114·589	38·1885	22·9038	16·3499	12·7062	10·3854	8·77689	7·59575	6·69116	30
31	110·892	37·7686	22·7519	16·2722	12·6591	10·3538	8·75425	7·57872	6·67787	29
32	107·426	37·3579	22·6020	16·1952	12·6124	10·3224	8·73172	7·56176	6·66463	28
33	104·171	36·9560	22·4541	16·1190	12·5660	10·2913	8·70931	7·54487	6·65144	27
34	101·107	36·5627	22·3081	16·0435	12·5199	10·2602	8·68701	7·52806	6·63831	26
35	98·2179	36·1776	22·1640	15·9687	12·4742	10·2294	8·66482	7·51132	6·62523	25
36	95·4895	35·8006	22·0217	15·8945	12·4288	10·1988	8·64275	7·49465	6·61219	24
37	92·9085	35·4313	21·8813	15·8211	12·3838	10·1683	8·62078	7·47806	6·59921	23
38	90·4633	35·0695	21·7426	15·7483	12·3390	10·1381	8·59893	7·46154	6·58627	22
39	88·1436	34·7151	21·6056	15·6762	12·2946	10·1080	8·57718	7·44509	6·57339	21
40	85·9398	34·3678	21·4704	15·6048	12·2505	10·0780	8·55555	7·42871	6·56055	20
41	83·8435	34·0273	21·3369	15·5340	12·2067	10·0483	8·53402	7·41240	6·54777	19
42	81·8470	33·6935	21·2049	15·4638	12·1632	10·0187	8·51259	7·39616	6·53503	18
43	79·9434	33·3662	21·0747	15·3943	12·1201	9·98931	8·49128	7·37999	6·52234	17
44	78·1263	33·0452	20·9460	15·3254	12·0772	9.96007	8.47007	7.36389	6·50970	16
45	76·3900	32·7303	20·8188	15·2571	12·0346	9·93101	8·44896	7·34786	6·49710	15
46	74·7292	32·4213	20·6932	15·1893	11·9923	9·90211	8·42795	7·33190	6·48456	14
47	73·1390	32·1181	20·5691	15·1222	11·9504	9·87338	8·40705	7·31600	6·47206	13
48	71·6151	31·8205	20·4465	15·0557	11·9087	9·84482	8·38625	7·30018	6·45961	12
49	70·1533	31·5284	20·3253	14·9898	11·8673	9·81641	8·36555	7·28442	6·44720	11
50	68·7501	31·2416	20·2056	14·9244	11·8262	9·78817	8·34496	7·26873	6·43484	10
51	67·4019	30·9599	20·0872	14·8596	11·7853	9·76009	8·32446	7·25310	6·42253	9
52	66·1055	30·6833	19·9702	14·7954	11·7448	9·73217	8·30406	7·23754	6·41026	8
53	64·8580	30·4116	19·8546	14·7317	11·7045	9·70441	8·28376	7·22204	6·39804	7
54	63·6567	30·1446	19·7403	14·6685	11·6645	9·67680	8·26355	7·20661	6·38587	6
55	62·4992	29·8823	19·6273	14·6059	11·6248	9·64935	8·24345	7·19125	6·37374	5
56	61·3829	29·6245	19·5156	14·5438	11·5853	9·62205	8·22344	7·17594	6·36165	4
57	60·3058	29·3711	19·4051	14·4823	11·5461	9·59490	8·20352	7·16071	6·34961	3
58	59·2659	29·1220	19·2959	14·4212	11·5072	9·56791	8·18370	7·14553	6·33761	2
59	58·2612	28·8771	19·1879	14·3607	11·4685	9·54106	8·16398	7·13042	6·32566	1
60	57·2900	28·6363	19·0811	14·3007	11·4301	9·51436	8·14435	7·11537	6·31375	0
′	89°	88°	87°	86°	85°	84°	83°	82°	81°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	9°	10°	11°	12°	13°	14°	15°	16°	17°	′
0	6·31375	5·67128	5·14455	4·70463	4·33148	4·01078	3·73205	3·48741	3·27085	60
1	6·30189	5·66165	5·13658	4·69791	4·32573	4·00582	3·72771	3·48359	3·26745	59
2	6·29007	5·65205	5·12862	4·69121	4·32001	4·00086	3·72338	3·47977	3·26406	58
3	6·27829	5·64248	5·12069	4·68452	4·31430	3·99592	3·71907	3·47596	3·26067	57
4	6·26655	5·63295	5·11279	4·67786	4·30860	3·99099	3·71476	3·47216	3·25729	56
5	6·25486	5·62344	5·10490	4·67121	4·30291	3·98607	3·71046	3·46837	3·25392	55
6	6·24321	5·61397	5·09704	4·66458	4·29724	3·98117	3·70616	3·46458	3·25055	54
7	6·23160	5·60452	5·08921	4·65797	4·29159	3·97627	3·70188	3·46080	3·24719	53
8	6·22003	5·59511	5·08139	4·65138	4·28595	3·97139	3·69761	3·45703	3·24383	52
9	6·20851	5·58573	5·07360	4·64480	4·28032	3·96651	3·69335	3·45327	3·24049	51
10	6·19703	5·57638	5·06584	4·63825	4·27471	3·96165	3·68909	3·44951	3·23714	50
11	6·18559	5·56706	5·05809	4·63171	4·26911	3·95680	3·68485	3·44576	3·23381	49
12	6·17419	5·55777	5·05037	4·62518	4·26352	3·95196	3·68061	3·44202	3·23048	48
13	6·16283	5·54851	5·04267	4·61868	4·25795	3·94713	3·67638	3·43829	3·22715	47
14	6·15151	5·53927	5·03499	4·61219	4·25239	3·94232	3·67217	3·43456	3·22384	46
15	6·14023	5·53007	5·02734	4·60572	4·24685	3·93751	3·66796	3·43084	3·22053	45
16	6·12899	5·52090	5·01971	4·59927	4·24132	3·93271	3·66376	3·42713	3·21722	44
17	6·11779	5·51176	5·01210	4·59283	4·23580	3·92793	3·65957	3·42343	3·21392	43
18	6·10664	5·50264	5·00451	4·58641	4·23030	3·92316	3·65538	3·41973	3·21063	42
19	6·09552	5·49356	4·99695	4·58001	4·22481	3·91839	3·65121	3·41604	3·20734	41
20	6·08444	5·48451	4·98940	4·57363	4·21933	3·91364	3·64705	3·41236	3·20406	40
21	6·07340	5·47548	4·98188	4·56726	4·21387	3·90890	3·64289	3·40869	3·20079	39
22	6·06240	5·46648	4·97438	4·56091	4·20842	3·90417	3·63874	3·40502	3·19752	38
23	6·05143	5·45751	4·96690	4·55458	4·20298	3·89945	3·63461	3·40136	3·19426	37
24	6·04051	5·44857	4·95945	4·54826	4·19756	3·89474	3·63048	3·39771	3·19100	36
25	6·02962	5·43966	4·95201	4·54196	4·19215	3·89004	3·62636	3·39406	3·18775	35
26	6·01878	5·43077	4·94460	4·53568	4·18675	3·88536	3·62224	3·39042	3·18451	34
27	6·00797	5·42192	4·93721	4·52941	4·18137	3·88068	3·61814	3·38679	3·18127	33
28	5·99720	5·41309	4·92984	4·52316	4·17600	3·87601	3·61405	3·38317	3·17804	32
29	5·98646	5·40429	4·92249	4·51693	4·17064	3·87136	3·60996	3·37955	3·17481	31
30	5·97576	5·39552	4·91516	4·51071	4·16530	3·86671	3·60588	3·37594	3·17159	30
31	5·96510	5·38677	4·90785	4·50451	4·15997	3·86208	3·60181	3·37234	3·16838	29
32	5·95448	5·37805	4·90056	4·49832	4·15465	3·85745	3·59775	3·36875	3·16517	28
33	5·94390	5·36936	4·89330	4·49215	4·14934	3·85284	3·59370	3·36516	3·16197	27
34	5·93335	5·36070	4·88605	4·48600	4·14405	3·84824	3·58966	3·36158	3·15877	26
35	5·92283	5·35206	4·87882	4·47986	4·13877	3·84364	3·58562	3·35800	3·15558	25
36	5·91236	5·34345	4·87162	4·47374	4·13350	3·83906	3·58160	3·35443	3·15240	24
37	5·90191	5·33487	4·86444	4·46764	4·12825	3·83449	3·57758	3·35087	3·14922	23
38	5·89151	5·32631	4·85727	4·46155	4·12301	3·82992	3·57357	3·34732	3·14605	22
39	5·88114	5·31778	4·85013	4·45548	4·11778	3·82537	3·56957	3·34377	3·14288	21
40	5·87080	5·30928	4·84300	4·44942	4·11256	3·82083	3·56557	3·34023	3·13972	20
41	5·86051	5·30080	4·83590	4·44338	4·10736	3·81630	3·56159	3·33670	3·13656	19
42	5·85024	5·29235	4·82882	4·43735	4·10216	3·81177	3·55761	3·33317	3·13341	18
43	5·84001	5·28393	4·82175	4·43134	4·09699	3·80726	3·55364	3·32965	3·13027	17
44	5·82982	5·27553	4·81471	4·42534	4·09182	3·80276	3·54968	3·32614	3·12713	16
45	5·81966	5·26715	4·80769	4·41936	4·08666	3·79827	3·54573	3·32264	3·12400	15
46	5·80953	5·25880	4·80068	4·41340	4·08152	3·79378	3·54179	3·31914	3·12087	14
47	5·79944	5·25048	4·79370	4·40745	4·07639	3·78931	3·53785	3·31565	3·11775	13
48	5·78938	5·24218	4·78673	4·40152	4·07127	3·78485	3·53393	3·31216	3·11464	12
49	5·77936	5·23391	4·77978	4·39560	4·06616	3·78040	3·53001	3·30868	3·11153	11
50	5·76937	5·22566	4·77286	4·38969	4·06107	3·77595	3·52609	3·30521	3·10842	10
51	5·75941	5·21744	4·76595	4·38381	4·05599	3·77152	3·52219	3·30174	3·10532	9
52	5·74949	5·20925	4·75906	4·37793	4·05092	3·76709	3·51829	3·29829	3·10223	8
53	5·73960	5·20107	4·75219	4·37207	4·04586	3·76268	3·51441	3·29483	3·09914	7
54	5·72974	5·19293	4·74534	4·36623	4·04081	3·75828	3·51053	3·29139	3·09606	6
55	5·71992	5·18480	4·73851	4·36040	4·03578	3·75388	3·50666	3·28795	3·09298	5
56	5·71013	5·17671	4·73170	4·35459	4·03076	3·74950	3·50279	3·28452	3·08991	4
57	5·70037	5·16863	4·72490	4·34879	4·02574	3·74512	3·49894	3·28109	3·08685	3
58	5·69064	5·16058	4·71813	4·34300	4·02074	3·74075	3·49509	3·27767	3·08379	2
59	5·68094	5·15256	4·71137	4·33723	4·01576	3·73640	3·49125	3·27426	3·08073	1
60	5·67128	5·14455	4·70463	4·33148	4·01078	3·73205	3·48741	3·27085	3·07768	0
′	80°	79°	78°	77°	76°	75°	74°	73°	72°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	18°	19°	20°	21°	22°	23°	24°	25°	26°	′
0	3·07768	2·90421	2·74748	2·60509	2·47509	2·35585	2·24604	2·14451	2·05030	60
1	3·07464	2·90147	2·74499	2·60283	2·47302	2·35395	2·24428	2·14288	2·04879	59
2	3·07160	2·89873	2·74251	2·60057	2·47095	2·35205	2·24252	2·14125	2·04728	58
3	3·06857	2·89600	2·74004	2·59831	2·46888	2·35015	2·24077	2·13963	2·04577	57
4	3·06554	2·89327	2·73756	2·59606	2·46682	2·34825	2·23902	2·13801	2·04426	56
5	3·06252	2·89055	2·73509	2·59381	2·46476	2·34636	2·23727	2·13639	2·04276	55
6	3·05950	2·88783	2·73263	2·59156	2·46270	2·34447	2·23553	2·13477	2·04125	54
7	3·05649	2·88511	2·73017	2·58932	2·46065	2·34258	2·23378	2·13316	2·03975	53
8	3·05349	2·88240	2·72771	2·58708	2·45860	2·34069	2·23204	2·13154	2·03825	52
9	3·05049	2·87970	2·72526	2·58484	2·45655	2·33881	2·23030	2·12993	2·03675	51
10	3·04749	2·87700	2·72281	2·58261	2·45451	2·33693	2·22857	2·12832	2·03526	50
11	3·04450	2·87430	2·72036	2·58038	2·45246	2·33505	2·22683	2·12671	2·03376	49
12	3·04152	2·87161	2·71792	2·57815	2·45043	2·33317	2·22510	2·12511	2·03227	48
13	3·03854	2·86892	2·71548	2·57593	2·44839	2·33130	2·22337	2·12350	2·03078	47
14	3·03556	2·86624	2·71305	2·57371	2·44636	2·32943	2·22164	2·12190	2·02929	46
15	3·03260	2·86356	2·71062	2·57150	2·44433	2·32756	2·21992	2·12030	2·02780	45
16	3·02963	2·86089	2·70819	2·56928	2·44230	2·32570	2·21819	2·11871	2·02631	44
17	3·02667	2·85822	2·70577	2·56707	2·44027	2·32383	2·21647	2·11711	2·02483	43
18	3·02372	2·85555	2·70335	2·56487	2·43825	2·32197	2·21475	2·11552	2·02335	42
19	3·02077	2·85289	2·70094	2·56266	2·43623	2·32012	2·21304	2·11392	2·02187	41
20	3·01783	2·85023	2·69853	2·56046	2·43422	2·31826	2·21132	2·11233	2·02039	40
21	3·01489	2·84758	2·69612	2·55827	2·43220	2·31641	2·20961	2·11075	2·01891	39
22	3·01196	2·84494	2·69371	2·55608	2·43019	2·31456	2·20790	2·10916	2·01743	38
23	3·00903	2·84229	2·69131	2·55389	2·42819	2·31271	2·20619	2·10758	2·01596	37
24	3·00611	2·83965	2·68892	2·55170	2·42618	2·31086	2·20449	2·10600	2·01449	36
25	3·00319	2·83702	2·68653	2·54952	2·42418	2·30902	2·20278	2·10442	2·01302	35
26	3·00028	2·83439	2·68414	2·54734	2·42218	2·30718	2·20108	2·10284	2·01155	34
27	2·99738	2·83176	2·68175	2·54516	2·42019	2·30534	2·19938	2·10126	2·01008	33
28	2·99447	2·82914	2·67937	2·54299	2·41819	2·30351	2·19769	2·09969	2·00862	32
29	2·99158	2·82653	2·67700	2·54082	2·41620	2·30167	2·19599	2·09811	2·00715	31
30	2·98868	2·82391	2·67462	2·53865	2·41421	2·29984	2·19430	2·09654	2·00569	30
31	2·98580	2·82130	2·67225	2·53648	2·41223	2·29801	2·19261	2·09498	2·00423	29
32	2·98292	2·81870	2·66989	2·53432	2·41025	2·29619	2·19092	2·09341	2·00277	28
33	2·98004	2·81610	2·66752	2·53217	2·40827	2·29437	2·18923	2·09184	2·00131	27
34	2·97717	2·81350	2·66516	2·53001	2·40629	2·29254	2·18755	2·09028	1·99986	26
35	2·97430	2·81091	2·66281	2·52786	2·40432	2·29073	2·18587	2·08872	1·99841	25
36	2·97144	2·80833	2·66046	2·52571	2·40235	2·28891	2·18419	2·08716	1·99695	24
37	2·96858	2·80574	2·65811	2·52357	2·40038	2·28710	2·18251	2·08560	1·99550	23
38	2·96573	2·80316	2·65576	2·52142	2·39841	2·28528	2·18084	2·08405	1·99406	22
39	2·96288	2·80059	2·65342	2·51929	2·39645	2·28348	2·17916	2·08250	1·99261	21
40	2·96004	2·79802	2·65109	2·51715	2·39449	2·28167	2·17749	2·08094	1·99116	20
41	2·95721	2·79545	2·64875	2·51502	2·39253	2·27987	2·17582	2·07939	1·98972	19
42	2·95437	2·79289	2·64642	2·51289	2·39058	2·27806	2·17416	2·07785	1·98828	18
43	2·95155	2·79033	2·64410	2·51076	2·38863	2·27626	2·17249	2·07630	1·98684	17
44	2·94872	2·78778	2·64177	2·50864	2·38668	2·27447	2·17083	2·07476	1·98540	16
45	2·94591	2·78523	2·63945	2·50652	2·38473	2·27267	2·16917	2·07321	1·98396	15
46	2·94309	2·78269	2·63714	2·50440	2·38279	2·27088	2·16751	2·07167	1·98253	14
47	2·94028	2·78014	2·63483	2·50229	2·38084	2·26909	2·16585	2·07014	1·98110	13
48	2·93748	2·77761	2·63252	2·50018	2·37891	2·26730	2·16420	2·06860	1·97966	12
49	2·93468	2·77507	2·63021	2·49807	2·37697	2·26552	2·16255	2·06706	1·97823	11
50	2·93189	2·77254	2·62791	2·49597	2·37504	2·26374	2·16090	2·06553	1·97681	10
51	2·92910	2·77002	2·62561	2·49386	2·37311	2·26196	2·15925	2·06400	1·97538	9
52	2·92632	2·76750	2·62332	2·49177	2·37118	2·26018	2·15760	2·06247	1·97395	8
53	2·92354	2·76498	2·62103	2·48967	2·36925	2·25840	2·15596	2·06094	1·97253	7
54	2·92076	2·76247	2·61874	2·48758	2·36733	2·25663	2·15432	2·05942	1·97111	6
55	2·91799	2·75996	2·61646	2·48549	2·36541	2·25486	2·15268	2·05790	1·96969	5
56	2·91523	2·75746	2·61418	2·48340	2·36349	2·25309	2·15104	2·05637	1·96827	4
57	2·91246	2·75496	2·61190	2·48132	2·36158	2·25132	2·14940	2·05485	1·96685	3
58	2·90971	2·75246	2·60963	2·47924	2·35967	2·24956	1·14777	2·05333	1·96544	2
59	2·90696	2·74997	2·60736	2·47716	2·35776	2·24780	2·14614	2·05182	1·96402	1
60	2·90421	2·74748	2·60509	2·47509	2·35585	2·24604	2·14451	2·05030	1·96261	0
′	71°	70°	69°	68°	67°	66°	65°	64°	63°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	27°	28°	29°	30°	31°	32°	33°	34°	35°	′
0	1·96261	1·88073	1·80405	1·73205	1·66428	1·60033	1·53986	1·48256	1·42815	60
1	1·96120	1·87941	1·80281	1·73089	1·66318	1·59930	1·53888	1·48163	1·42726	59
2	1·95979	1·87809	1·80158	1·72973	1·66209	1·59826	1·53791	1·48070	1·42638	58
3	1·95838	1·87677	1·80034	1·72857	1·66099	1·59723	1·53693	1·47977	1·42550	57
4	1·95698	1·87546	1·79911	1·72741	1·65990	1·59620	1·53595	1·47885	1·42462	56
5	1·95557	1·87415	1·79788	1·72625	1·65881	1·59517	1·53497	1·47792	1·42374	55
6	1·95417	1·87283	1·79665	1·72509	1·65772	1·59414	1·53400	1·47699	1·42286	54
7	1·95277	1·87152	1·79542	1·72393	1·65663	1·59311	1·53302	1·47607	1·42198	53
8	1·95137	1·87021	1·79419	1·72278	1·65554	1·59208	1·53205	1·47514	1·42110	52
9	1·94997	1·86891	1·79296	1·72163	1·65445	1·59105	1·53107	1·47422	1·42022	51
10	1·94858	1·86760	1·79174	1·72047	1·65337	1·59002	1·53010	1·47330	1·41934	50
11	1·94718	1·86630	1·79051	1·71932	1·65228	1·58900	1·52913	1·47238	1·41847	49
12	1·94579	1·86499	1·78929	1·71817	1·65120	1·58797	1·52816	1·47146	1·41759	48
13	1·94440	1·86369	1·78807	1·71702	1·65011	1·58695	1·52719	1·47053	1·41672	47
14	1·94301	1·86239	1·78685	1·71588	1·64903	1·58593	1·52622	1·46962	1·41584	46
15	1·94162	1·86109	1·78563	1·71473	1·64795	1·58490	1·52525	1·46870	1·41497	45
16	1·94023	1·85979	1·78441	1·71358	1·64687	1·58388	1·52429	1·46778	1·41409	44
17	1·93885	1·85850	1·78319	1·71244	1·64579	1·58286	1·52332	1·46686	1·41322	43
18	1·93746	1·85720	1·78198	1·71129	1·64471	1·58184	1·52235	1·46595	1·41235	42
19	1·93608	1·85591	1·78077	1·71015	1·64363	1·58083	1·52139	1·46503	1·41148	41
20	1·93470	1·85462	1·77955	1·70901	1·64256	1·57981	1·52043	1·46411	1·41061	40
21	1·93332	1·85333	1·77834	1·70787	1·64148	1·57879	1·51946	1·46320	1·40974	39
22	1·93195	1·85204	1·77713	1·70673	1·64041	1·57778	1·51850	1·46229	1·40887	38
23	1·93057	1·85075	1·77592	1·70560	1·63934	1·57676	1·51754	1·46137	1·40800	37
24	1·92920	1·84946	1·77471	1·70446	1·63826	1·57575	1·51658	1·46046	1·40714	36
25	1·92782	1·84818	1·77351	1·70332	1·63719	1·57474	1·51562	1·45955	1·40627	35
26	1·92645	1·84689	1·77230	1·70219	1·63612	1·57372	1·51466	1·45864	1·40540	34
27	1·92508	1·84561	1·77110	1·70106	1·63505	1·57271	1·51370	1·45773	1·40454	33
28	1·92371	1·84433	1·76990	1·69992	1·63398	1·57170	1·51275	1·45682	1·40367	32
29	1·92235	1·84305	1·76869	1·69879	1·63292	1·57069	1·51179	1·45592	1·40281	31
30	1·92098	1·84177	1·76749	1·69766	1·63185	1·56969	1·51084	1·45501	1·40195	30
31	1·91962	1·84049	1·76630	1·69653	1·63079	1·56868	1·50988	1·45410	1·40109	29
32	1·91826	1·83922	1·76510	1·69541	1·62972	1·56767	1·50893	1·45320	1·40022	28
33	1·91690	1·83794	1·76390	1·69428	1·62866	1·56667	1·50797	1·45229	1·39936	27
34	1·91554	1·83667	1·76271	1·69316	1·62760	1·56566	1·50702	1·45139	1·39850	26
35	1·91418	1·83540	1·76151	1·69203	1·62654	1·56466	1·50607	1·45049	1·39764	25
36	1·91282	1·83413	1·76032	1·69091	1·62548	1·56366	1·50512	1·44958	1·39679	24
37	1·91147	1·83286	1·75913	1·68979	1·62442	1·56265	1·50417	1·44868	1·39593	23
38	1·91012	1·83159	1·75794	1·68866	1·62336	1·56165	1·50322	1·44778	1·39507	22
39	1·90876	1·83033	1·75675	1·68754	1·62230	1·56065	1·50228	1·44688	1·39421	21
40	1·90741	1·82906	1·75556	1·68643	1·62125	1·55966	1·50133	1·44598	1·39336	20
41	1·90607	1·82780	1·75437	1·68531	1·62019	1·55866	1·50038	1·44508	1·39250	19
42	1·90472	1·82654	1·75319	1·68419	1·61914	1·55766	1·49944	1·44418	1·39165	18
43	1·90337	1·82528	1·75200	1·68308	1·61808	1·55666	1·49849	1·44329	1·39079	17
44	1·90203	1·82402	1·75082	1·68196	1·61703	1·55567	1·49755	1·44239	1·38994	16
45	1·90069	1·82276	1·74964	1·68085	1·61598	1·55467	1·49661	1·44149	1·38909	15
46	1·89935	1·82150	1·74846	1·67974	1·61493	1·55368	1·49566	1·44060	1·38824	14
47	1·89801	1·82025	1·74728	1·67863	1·61388	1·55269	1·49472	1·43970	1·38738	13
48	1·89667	1·81899	1·74610	1·67752	1·61283	1·55170	1·49378	1·43881	1·38653	12
49	1·89533	1·81774	1·74492	1·67641	1·61179	1·55071	1·49284	1·43792	1·38568	11
50	1·89400	1·81649	1·74375	1·67530	1·61074	1·54972	1·49190	1·43703	1·38484	10
51	1·89266	1·81524	1·74257	1·67419	1·60970	1·54873	1·49097	1·43614	1·38399	9
52	1·89133	1·81399	1·74140	1·67309	1·60865	1·54774	1·49003	1·43525	1·38314	8
53	1·89000	1·81274	1·74022	1·67198	1·60761	1·54675	1·48909	1·43436	1·38229	7
54	1·88867	1·81150	1·73905	1·67088	1·60657	1·54576	1·48816	1·43347	1·38145	6
55	1·88734	1·81025	1·73788	1·66978	1·60553	1·54478	1·48722	1·43258	1·38060	5
56	1·88602	1·80901	1·73671	1·66867	1·60449	1·54379	1·48629	1·43169	1·37976	4
57	1·88469	1·80777	1·73555	1·66757	1·60345	1·54281	1·48536	1·43080	1·37891	3
58	1·88337	1·80653	1·73438	1·66647	1·60241	1·54183	1·48442	1·42992	1·37807	2
59	1·88205	1·80529	1·73321	1·66538	1·60137	1·54085	1·48349	1·42903	1·37722	1
60	1·88073	1·80405	1·73205	1·66428	1·60033	1·53986	1·48256	1·42815	1·37638	0
′	62°	61°	60°	59°	58°	57°	56°	55°	54°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	36°	37°	38°	39°	40°	41°	42°	43°	44°	′
0	1·37638	1·32704	1·27994	1·23490	1·19175	1·15037	1·11061	1·07237	1·03553	60
1	1·37554	1·32624	1·27917	1·23416	1·19105	1·14969	1·10996	1·07174	1·03493	59
2	1·37470	1·32544	1·27841	1·23343	1·19035	1·14902	1·10931	1·07112	1·03433	58
3	1·37386	1·32464	1·27764	1·23270	1·18964	1·14834	1·10867	1·07049	1·03372	57
4	1·37302	1·32384	1·27688	1·23196	1·18894	1·14767	1·10802	1·06987	1·03312	56
5	1·37218	1·32304	1·27611	1·23123	1·18824	1·14699	1·10737	1·06925	1·03252	55
6	1·37134	1·32224	1·27535	1·23050	1·18754	1·14632	1·10672	1·06862	1·03192	54
7	1·37050	1·32144	1·27458	1·22977	1·18684	1·14565	1·10607	1·06800	1·03132	53
8	1·36967	1·32064	1·27382	1·22904	1·18614	1·14498	1·10543	1·06738	1·03072	52
9	1·36883	1·31984	1·27306	1·22831	1·18544	1·14430	1·10478	1·06676	1·03012	51
10	1·36800	1·31904	1·27230	1·22758	1·18474	1·14363	1·10414	1·06613	1·02952	50
11	1·36716	1·31825	1·27153	1·22685	1·18404	1·14296	1·10349	1·06551	1·02892	49
12	1·36633	1·31745	1·27077	1·22612	1·18334	1·14229	1·10285	1·06489	1·02832	48
13	1·36549	1·31666	1·27001	1·22539	1·18264	1·14162	1·10220	1·06427	1·02772	47
14	1·36466	1·31586	1·26925	1·22467	1·18194	1·14095	1·10156	1·06365	1·02713	46
15	1·36383	1·31507	1·26849	1·22394	1·18125	1·14028	1·10091	1·06303	1·02653	45
16	1·36300	1·31427	1·26774	1·22321	1·18055	1·13961	1·10027	1·06241	1·02593	44
17	1·36217	1·31348	1·26698	1·22249	1·17986	1·13894	1·09963	1·06179	1·02533	43
18	1·36134	1·31269	1·26622	1·22176	1·17916	1·13828	1·09899	1·06117	1·02474	42
19	1·36051	1·31190	1·26546	1·22104	1·17846	1·13761	1·09834	1·06056	1·02414	41
20	1·35968	1·31110	1·26471	1·22031	1·17777	1·13694	1·09770	1·05994	1·02355	40
21	1·35885	1·31031	1·26395	1·21959	1·17708	1·13627	1·09706	1·05932	1·02295	39
22	1·35802	1·30952	1·26319	1·21886	1·17638	1·13561	1·09642	1·05870	1·02236	38
23	1·35719	1·30873	1·26244	1·21814	1·17569	1·13494	1·09578	1·05809	1·02176	37
24	1·35637	1·30795	1·26169	1·21742	1·17500	1·13428	1·09514	1·05747	1·02117	36
25	1·35554	1·30716	1·26093	1·21670	1·17430	1·13361	1·09450	1·05685	1·02057	35
26	1·35472	1·30637	1·26018	1·21598	1·17361	1·13295	1·09386	1·05624	1·01998	34
27	1·35389	1·30558	1·25943	1·21526	1·17292	1·13228	1·09322	1·05562	1·01939	33
28	1·35307	1·30480	1·25867	1·21454	1·17223	1·13162	1·09258	1·05501	1·01879	32
29	1·35224	1·30401	1·25792	1·21382	1·17154	1·13096	1·09195	1·05439	1·01820	31
30	1·35142	1·30323	1·25717	1·21310	1·17085	1·13029	1·09131	1·05378	1·01761	30
31	1·35060	1·30244	1·25642	1·21238	1·17016	1·12963	1·09067	1·05317	1·01702	29
32	1·34978	1·30166	1·25567	1·21166	1·16947	1·12897	1·09003	1·05255	1·01642	28
33	1·34896	1·30087	1·25492	1·21094	1·16878	1·12831	1·08940	1·05194	1·01583	27
34	1·34814	1·30009	1·25417	1·21023	1·16809	1·12765	1·08876	1·05133	1·01524	26
35	1·34732	1·29931	1·25343	1·20951	1·16741	1·12699	1·08813	1·05072	1·01465	25
36	1·34650	1·29853	1·25268	1·20879	1·16672	1·12633	1·08749	1·05010	1·01406	24
37	1·34568	1·29775	1·25193	1·20808	1·16603	1·12567	1·08686	1·04949	1·01347	23
38	1·34487	1·29696	1·25118	1·20736	1·16535	1·12501	1·08622	1·04888	1·01288	22
39	1·34405	1·29618	1·25044	1·20665	1·16466	1·12435	1·08559	1·04827	1·01229	21
40	1·34323	1·29541	1·24969	1·20593	1·16398	1·12369	1·08496	1·04766	1·01170	20
41	1·34242	1·29463	1·24895	1·20522	1·16329	1·12303	1·08432	1·04705	1·01112	19
42	1·34160	1·29385	1·24820	1·20451	1·16261	1·12238	1·08369	1·04644	1·01053	18
43	1·34079	1·29307	1·24746	1·20379	1·16192	1·12172	1·08306	1·04583	1·00994	17
44	1·33998	1·29229	1·24672	1·20308	1·16124	1·12106	1·08243	1·04522	1·00935	16
45	1·33916	1·29152	1·24597	1·20237	1·16056	1·12041	1·08179	1·04461	1·00876	15
46	1·33835	1·29074	1·24523	1·20166	1·15987	1·11975	1·08116	1·04401	1·00818	14
47	1·33754	1·28997	1·24449	1·20095	1·15919	1·11909	1·08053	1·04340	1·00759	13
48	1·33673	1·28919	1·24375	1·20024	1·15851	1·11844	1·07990	1·04279	1·00701	12
49	1·33592	1·28842	1·24301	1·19953	1·15783	1·11778	1·07927	1·04218	1·00642	11
50	1·33511	1·28764	1·24227	1·19882	1·15715	1·11713	1·07864	1·04158	1·00583	10
51	1·33430	1·28687	1·24153	1·19811	1·15647	1·11648	1·07801	1·04097	1·00525	9
52	1·33349	1·28610	1·24079	1·19740	1·15579	1·11582	1·07738	1·04036	1·00467	8
53	1·33268	1·28533	1·24005	1·19669	1·15511	1·11517	1·07676	1·03976	1·00408	7
54	1·33187	1·28456	1·23931	1·19599	1·15443	1·11452	1·07613	1·03915	1·00350	6
55	1·33107	1·28379	1·23858	1·19528	1·15375	1·11387	1·07550	1·03855	1·00291	5
56	1·33026	1·28302	1·23784	1·19457	1·15308	1·11321	1·07487	1·03794	1·00233	4
57	1·32946	1·28225	1·23710	1·19387	1·15240	1·11256	1·07425	1·03734	1·00175	3
58	1·32865	1·28148	1·23637	1·19316	1·15172	1·11191	1·07362	1·03674	1·00116	2
59	1·32785	1·28071	1·23563	1·19246	1·15104	1·11126	1·07299	1·03613	1·00058	1
60	1·32704	1·27994	1·23490	1·19175	1·15037	1·11061	1·07237	1·03553	1·00000	0
′	53°	52°	51°	50°	49°	48°	47°	46°	45°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	45°	46°	47°	48°	49°	50°	51°	52°	53°	′
0	1·00000	0·96569	0·93252	0·90040	0·86929	0·83910	0·80978	0·78129	0·75355	60
1	0·99942	0·96513	0·93197	0·89988	0·86878	0·83860	0·80930	0·78082	0·75310	59
2	0·99884	0·96457	0·93143	0·89935	0·86827	0·83811	0·80882	0·78035	0·75264	58
3	0·99826	0·96400	0·93088	0·89883	0·86776	0·83761	0·80834	0·77988	0·75219	57
4	0·99768	0·96344	0·93034	0·89830	0·86725	0·83712	0·80786	0·77941	0·75173	56
5	0·99710	0·96288	0·92980	0·89777	0·86674	0·83662	0·80738	0·77895	0·75128	55
6	0·99652	0·96232	0·92926	0·89725	0·86623	0·83613	0·80690	0·77848	0·75082	54
7	0·99594	0·96176	0·92872	0·89672	0·86572	0·83564	0·80642	0·77801	0·75037	53
8	0·99536	0·96120	0·92817	0·89620	0·86521	0·83514	0·80594	0·77754	0·74991	52
9	0·99478	0·96064	0·92763	0·89567	0·86470	0·83465	0·80546	0·77708	0·74946	51
10	0·99420	0·96008	0·92709	0·89515	0·86419	0·83415	0·80498	0·77661	0·74900	50
11	0·99362	0·95952	0·92655	0·89463	0·86368	0·83366	0·80450	0·77615	0·74855	49
12	0·99304	0·95897	0·92601	0·89410	0·86318	0·83317	0·80402	0·77568	0·74810	48
13	0·99247	0·95841	0·92547	0·89358	0·86267	0·83268	0·80354	0·77521	0·74764	47
14	0·99189	0·95785	0·92493	0·89306	0·86216	0·83218	0·80306	0·77475	0·74719	46
15	0·99131	0·95729	0·92439	0·89253	0·86166	0·83169	0·80258	0·77428	0·74674	45
16	0·99073	0·95673	0·92385	0·89201	0·86115	0·83120	0·80211	0·77382	0·74628	44
17	0·99016	0·95618	0·92331	0·89149	0·86064	0·83071	0·80163	0·77335	0·74583	43
18	0·98958	0·95562	0·92277	0·89097	0·86014	0·83022	0·80115	0·77289	0·74538	42
19	0·98901	0·95506	0·92224	0·89045	0·85963	0·82972	0·80067	0·77242	0·74492	41
20	0·98843	0·95451	0·92170	0·88992	0·85912	0·82923	0·80020	0·77196	0·74447	40
21	0·98786	0·95395	0·92116	0·88940	0·85862	0·82874	0·79972	0·77149	0·74402	39
22	0·98728	0·95340	0·92062	0·88888	0·85811	0·82825	0·79924	0·77103	0·74357	38
23	0·98671	0·95284	0·92008	0·88836	0·85761	0·82776	0·79877	0·77057	0·74312	37
24	0·98613	0·95229	0·91955	0·88784	0·85710	0·82727	0·79829	0·77010	0·74267	36
25	0·98556	0·95173	0·91901	0·88732	0·85660	0·82678	0·79781	0·76964	0·74221	35
26	0·98499	0·95118	0·91847	0·88680	0·85609	0·82629	0·79734	0·76918	0·74176	34
27	0·98441	0·95062	0·91794	0·88628	0·85559	0·82580	0·79686	0·76871	0·74131	33
28	0·98384	0·95007	0·91740	0·88576	0·85509	0·82531	0·79639	0·76825	0·74086	32
29	0·98327	0·94952	0·91687	0·88524	0·85458	0·82483	0·79591	0·76779	0·74041	31
30	0·98270	0·94896	0·91633	0·88473	0·85408	0·82434	0·79544	0·76733	0·73996	30
31	0·98213	0·94841	0·91580	0·88421	0·85358	0·82385	0·79496	0·76686	0·73951	29
32	0·98155	0·94786	0·91526	0·88369	0·85308	0·82336	0·79449	0·76640	0·73906	28
33	0·98098	0·94731	0·91473	0·88317	0·85257	0·82287	0·79401	0·76594	0·73861	27
34	0·98041	0·94676	0·91419	0·88265	0·85207	0·82238	0·79354	0·76548	0·73816	26
35	0·97984	0·94620	0·91366	0·88214	0·85157	0·82190	0·79306	0·76502	0·73771	25
36	0·97927	0·94565	0·91313	0·88162	0·85107	0·82141	0·79259	0·76456	0·73726	24
37	0·97870	0·94510	0·91259	0·88110	0·85057	0·82092	0·79212	0·76410	0·73681	23
38	0·97813	0·94455	0·91206	0·88059	0·85006	0·82044	0·79164	0·76364	0·73637	22
39	0·97756	0·94400	0·91153	0·88007	0·84956	0·81995	0·79117	0·76318	0·73592	21
40	0·97700	0·94345	0·91099	0·87955	0·84906	0·81946	0·79070	0·76272	0·73547	20
41	0·97643	0·94290	0·91046	0·87904	0·84856	0·81898	0·79022	0·76226	0·73502	19
42	0·97586	0·94235	0·90993	0·87852	0·84806	0·81849	0·78975	0·76180	0·73457	18
43	0·97529	0·94180	0·90940	0·87801	0·84756	0·81800	0·78928	0·76134	0·73413	17
44	0·97472	0·94125	0·90887	0·87749	0·84706	0·81752	0·78881	0·76088	0·73368	16
45	0·97416	0·94071	0·90834	0·87698	0·84656	0·81703	0·78834	0·76042	0·73323	15
46	0·97359	0·94016	0·90781	0·87646	0·84606	0·81655	0·78786	0·75996	0·73278	14
47	0·97302	0·93961	0·90727	0·87595	0·84556	0·81606	0·78739	0·75950	0·73234	13
48	0·97246	0·93906	0·90674	0·87543	0·84507	0·81558	0·78692	0·75904	0·73189	12
49	0·97189	0·93852	0·90621	0·87492	0·84457	0·81510	0·78645	0·75858	0·73144	11
50	0·97133	0·93797	0·90569	0·87441	0·84407	0·81461	0·78598	0·75812	0·73100	10
51	0·97076	0·93742	0·90516	0·87389	0·84357	0·81413	0·78551	0·75767	0·73055	9
52	0·97020	0·93688	0·90463	0·87338	0·84307	0·81364	0·78504	0·75721	0·73010	8
53	0·96963	0·93633	0·90410	0·87287	0·84258	0·81316	0·78457	0·75675	0·72966	7
54	0·96907	0·93578	0·90357	0·87236	0·84208	0·81268	0·78410	0·75629	0·72921	6
55	0·96850	0·93524	0·90304	0·87184	0·84158	0·81220	0·78363	0·75584	0·72877	5
56	0·96794	0·93469	0·90251	0·87133	0·84108	0·81171	0·78316	0·75538	0·72832	4
57	0·96738	0·93415	0·90199	0·87082	0·84059	0·81123	0·78269	0·75492	0·72788	3
58	0·96681	0·93360	0·90146	0·87031	0·84009	0·81075	0·78222	0·75447	0·72743	2
59	0·96625	0·93306	0·90093	0·86980	0·83960	0·81027	0·78175	0·75401	0·72699	1
60	0·96569	0·93252	0·90040	0·86929	0·83910	0·80978	0·78129	0·75355	0·72654	0
′	44°	43°	42°	41°	40°	39°	38°	37°	36°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	54°	55°	56°	57°	58°	59°	60°	61°	62°	′
0	0·72654	0·70021	0·67451	0·64941	0·62487	0·60086	0·57735	0·55431	0·53171	60
1	0·72610	0·69977	0·67409	0·64899	0·62446	0·60046	0·57696	0·55393	0·53134	59
2	0·72565	0·69934	0·67366	0·64858	0·62406	0·60007	0·57657	0·55355	0·53096	58
3	0·72521	0·69891	0·67324	0·64817	0·62366	0·59967	0·57619	0·55317	0·53059	57
4	0·72477	0·69847	0·67282	0·64775	0·62325	0·59928	0·57580	0·55279	0·53022	56
5	0·72432	0·69804	0·67239	0·64734	0·62285	0·59888	0·57541	0·55241	0·52985	55
6	0·72388	0·69761	0·67197	0·64693	0·62245	0·59849	0·57503	0·55203	0·52947	54
7	0·72344	0·69718	0·67155	0·64652	0·62204	0·59809	0·57464	0·55165	0·52910	53
8	0·72299	0·69675	0·67113	0·64610	0·62164	0·59770	0·57425	0·55127	0·52873	52
9	0·72255	0·69631	0·67071	0·64569	0·62124	0·59730	0·57386	0·55089	0·52836	51
10	0·72211	0·69588	0·67028	0·64528	0·62083	0·59691	0·57348	0·55051	0·52798	50
11	0·72167	0·69545	0·66986	0·64487	0·62043	0·59651	0·57309	0·55013	0·52761	49
12	0·72122	0·69502	0·66944	0·64446	0·62003	0·59612	0·57271	0·54975	0·52724	48
13	0·72078	0·69459	0·66902	0·64404	0·61962	0·59573	0·57232	0·54938	0·52687	47
14	0·72034	0·69416	0·66860	0·64363	0·61922	0·59533	0·57193	0·54900	0·52650	46
15	0·71990	0·69372	0·66818	0·64322	0·61882	0·59494	0·57155	0·54862	0·52613	45
16	0·71946	0·69329	0·66776	0·64281	0·61842	0·59454	0·57116	0·54824	0·52575	44
17	0·71901	0·69286	0·66734	0·64240	0·61801	0·59415	0·57078	0·54786	0·52538	43
18	0·71857	0·69243	0·66692	0·64199	0·61761	0·59376	0·57039	0·54748	0·52501	42
19	0·71813	0·69200	0·66650	0·64158	0·61721	0·59336	0·57000	0·54711	0·52464	41
20	0·71769	0·69157	0·66608	0·64117	0·61681	0·59297	0·56962	0·54673	0·52427	40
21	0·71725	0·69114	0·66566	0·64076	0·61641	0·59258	0·56923	0·54635	0·52390	39
22	0·71681	0·69071	0·66524	0·64035	0·61601	0·59218	0·56885	0·54597	0·52353	38
23	0·71637	0·69028	0·66482	0·63994	0·61561	0·59179	0·56846	0·54560	0·52316	37
24	0·71593	0·68985	0·66440	0·63953	0·61520	0·59140	0·56808	0·54522	0·52279	36
25	0·71549	0·68942	0·66398	0·63912	0·61480	0·59101	0·56769	0·54484	0·52242	35
26	0·71505	0·68900	0·66356	0·63871	0·61440	0·59061	0·56731	0·54446	0·52205	34
27	0·71461	0·68857	0·66314	0·63830	0·61400	0·59022	0·56693	0·54409	0·52168	33
28	0·71417	0·68814	0·66272	0·63789	0·61360	0·58983	0·56654	0·54371	0·52131	32
29	0·71373	0·68771	0·66230	0·63748	0·61320	0·58944	0·56616	0·54333	0·52094	31
30	0·71329	0·68728	0·66189	0·63707	0·61280	0·58905	0·56577	0·54296	0·52057	30
31	0·71285	0·68685	0·66147	0·63666	0·61240	0·58865	0·56539	0·54258	0·52020	29
32	0·71242	0·68642	0·66105	0·63625	0·61200	0·58826	0·56501	0·54220	0·51983	28
33	0·71198	0·68600	0·66063	0·63584	0·61160	0·58787	0·56462	0·54183	0·51946	27
34	0·71154	0·68557	0·66021	0·63544	0·61120	0·58748	0·56424	0·54145	0·51909	26
35	0·71110	0·68514	0·65980	0·63503	0·61080	0·58709	0·56385	0·54107	0·51872	25
36	0·71066	0·68471	0·65938	0·63462	0·61040	0·58670	0·56347	0·54070	0·51835	24
37	0·71023	0·68429	0·65896	0·63421	0·61000	0·58631	0·56309	0·54032	0·51798	23
38	0·70979	0·68386	0·65854	0·63380	0·60960	0·58591	0·56270	0·53995	0·51761	22
39	0·70935	0·68343	0·65813	0·63340	0·60921	0·58552	0·56232	0·53957	0·51724	21
40	0·70891	0·68301	0·65771	0·63299	0·60881	0·58513	0·56194	0·53920	0·51688	20
41	0·70848	0·68258	0·65729	0·63258	0·60841	0·58474	0·56156	0·53882	0·51651	19
42	0·70804	0·68215	0·65688	0·63217	0·60801	0·58435	0·56117	0·53844	0·51614	18
43	0·70760	0·68173	0·65646	0·63177	0·60761	0·58396	0·56079	0·53807	0·51577	17
44	0·70717	0·68130	0·65604	0·63136	0·60721	0·58357	0·56041	0·53769	0·51540	16
45	0·70673	0·68088	0·65563	0·63095	0·60681	0·58318	0·56003	0·53732	0·51503	15
46	0·70629	0·68045	0·65521	0·63055	0·60642	0·58279	0·55964	0·53694	0·51467	14
47	0·70586	0·68002	0·65480	0·63014	0·60602	0·58240	0·55926	0·53657	0·51430	13
48	0·70542	0·67960	0·65438	0·62973	0·60562	0·58201	0·55888	0·53620	0·51393	12
49	0·70499	0·67917	0·65397	0·62933	0·60522	0·58162	0·55850	0·53582	0·51356	11
50	0·70455	0·67875	0·65355	0·62892	0·60483	0·58124	0·55812	0·53545	0·51319	10
51	0·70412	0·67832	0·65314	0·62852	0·60443	0·58085	0·55774	0·53507	0·51283	9
52	0·70368	0·67790	0·65272	0·62811	0·60403	0·58046	0·55736	0·53470	0·51246	8
53	0·70325	0·67748	0·65231	0·62770	0·60364	0·58007	0·55697	0·53432	0·51209	7
54	0·70281	0·67705	0·65189	0·62730	0·60324	0·57968	0·55659	0·53395	0·51173	6
55	0·70238	0·67663	0·65148	0·62689	0·60284	0·57929	0·55621	0·53358	0·51136	5
56	0·70194	0·67620	0·65106	0·62649	0·60245	0·57890	0·55583	0·53320	0·51099	4
57	0·70151	0·67578	0·65065	0·62608	0·60205	0·57851	0·55545	0·53283	0·51063	3
58	0·70107	0·67536	0·65024	0·62568	0·60165	0·57813	0·55507	0·53246	0·51026	2
59	0·70064	0·67493	0·64982	0·62527	0·60126	0·57774	0·55469	0·53208	0·50989	1
60	0·70021	0·67451	0·64941	0·62487	0·60086	0·57735	0·55431	0·53171	0·50953	0
′	35°	34°	33°	32°	31°	30°	29°	28°	27°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	63°	64°	65°	66°	67°	68°	69°	70°	71°	′
0	0·50953	0·48773	0·46631	0·44523	0·42447	0·40403	0·38386	0·36397	0·34433	60
1	0·50916	0·48737	0·46595	0·44488	0·42413	0·40369	0·38353	0·36364	0·34400	59
2	0·50879	0·48701	0·46560	0·44453	0·42379	0·40335	0·38320	0·36331	0·34368	58
3	0·50843	0·48665	0·46525	0·44418	0·42345	0·40301	0·38286	0·36298	0·34335	57
4	0·50806	0·48629	0·46489	0·44384	0·42310	0·40267	0·38253	0·36265	0·34303	56
5	0·50769	0·48593	0·46454	0·44349	0·42276	0·40234	0·38220	0·36232	0·34270	55
6	0·50733	0·48557	0·46418	0.44314	0.42242	0·40200	0·38186	0·36199	0·34238	54
7	0·50696	0·48521	0.46383	0.44279	0.42207	0·40166	0·38153	0·36167	0·34205	53
8	0·50660	0·48486	0·46348	0·44244	0·42173	0·40132	0·38120	0·36134	0·34173	52
9	0·50623	0·48450	0·46312	0·44210	0·42139	0·40098	0·38086	0·36101	0·34140	51
10	0·50587	0·48414	0·46277	0·44175	0·42105	0·40065	0·38053	0·36068	0·34108	50
11	0·50550	0·48378	0·46242	0·44140	0·42070	0·40031	0·38020	0·36035	0·34075	49
12	0·50514	0·48342	0·46206	0·44105	0·42036	0·39997	0·37986	0·36002	0·34043	48
13	0·50477	0·48306	0·46171	0·44071	0·42002	0·39963	0·37953	0·35969	0·34010	47
14	0·50441	0·48270	0·46136	0·44036	0·41968	0·39930	0·37920	0·35937	0·33978	46
15	0·50404	0·48234	0·46101	0·44001	0·41933	0·39896	0·37887	0·35904	0·33945	45
16	0·50368	0·48198	0·46065	0·43966	0·41899	0·39862	0·37853	0·35871	0·33913	44
17	0·50331	0·48163	0·46030	0·43932	0·41865	0·39829	0·37820	0·35838	0·33881	43
18	0·50295	0·48127	0·45995	0·43897	0·41831	0·39795	0·37787	0·35805	0·33848	42
19	0·50258	0·48091	0·45960	0·43862	0·41797	0·39761	0·37754	0·35772	0·33816	41
20	0·50222	0·48055	0·45924	0·43828	0·41763	0·39727	0·37720	0·35740	0·33783	40
21	0·50185	0·48019	0·45889	0·43793	0·41728	0·39694	0·37687	0·35707	0·33751	39
22	0·50149	0·47984	0·45854	0·43758	0·41694	0·39660	0·37654	0·35674	0·33718	38
23	0·50113	0·47948	0·45819	0·43724	0·41660	0·39626	0·37621	0·35641	0·33686	37
24	0·50076	0·47912	0·45784	0·43689	0·41626	0·39593	0·37588	0·35608	0·33654	36
25	0·50040	0·47876	0·45748	0·43654	0·41592	0·39559	0·37554	0·35576	0·33621	35
26	0·50004	0·47840	0·45713	0·43620	0·41558	0·39526	0·37521	0·35543	0·33589	34
27	0·49967	0·47805	0·45678	0·43585	0·41524	0·39492	0·37488	0·35510	0·33557	33
28	0·49931	0·47769	0·45643	0·43550	0·41490	0·39458	0·37455	0·35477	0·33524	32
29	0·49894	0·47733	0·45608	0·43516	0·41455	0·39425	0·37422	0·35445	0·33492	31
30	0·49858	0·47698	0·45573	0·43481	0·41421	0·39391	0·37388	0·35412	0·33460	30
31	0·49822	0·47662	0·45538	0·43447	0·41387	0·39357	0·37355	0·35379	0·33427	29
32	0·49786	0·47626	0·45502	0·43412	0·41353	0·39324	0·37322	0·35346	0·33395	28
33	0·49749	0·47590	0·45467	0·43378	0·41319	0·39290	0·37289	0·35314	0·33363	27
34	0·49713	0·47555	0·45432	0·43343	0·41285	0·39257	0·37256	0·35281	0·33330	26
35	0·49677	0·47519	0·45397	0·43308	0·41251	0·39223	0·37223	0·35248	0·33298	25
36	0·49640	0·47483	0·45362	0·43274	0·41217	0·39190	0·37190	0·35216	0·33266	24
37	0·49604	0·47448	0·45327	0·43239	0·41183	0·39156	0·37157	0·35183	0·33233	23
38	0·49568	0·47412	0·45292	0·43205	0·41149	0·39122	0·37123	0·35150	0·33201	22
39	0·49532	0·47377	0·45257	0·43170	0·41115	0·39089	0·37090	0·35118	0·33169	21
40	0·49495	0·47341	0·45222	0·43136	0·41081	0·39055	0·37057	0·35085	0·33136	20
41	0.49459	0·47305	0·45187	0·43101	0·41047	0·39022	0·37024	0·35052	0·33104	19
42	0·49423	0·47270	0·45152	0·43067	0·41013	0·38988	0·36991	0·35020	0·33072	18
43	0·49387	0·47234	0·45117	0·43032	0·40979	0·38955	0·36958	0·34987	0·33040	17
44	0·49351	0·47199	0·45082	0·42998	0·40945	0·38921	0·36925	0·34954	0·33007	16
45	0·49315	0·47163	0·45047	0·42963	0·40911	0·38888	0·36892	0·34922	0·32975	15
46	0·49278	0·47128	0·45012	0·42929	0·40877	0·38854	0·36859	0·34889	0·32943	14
47	0·49242	0·47092	0·44977	0·42894	0·40843	0·38821	0·36826	0·34856	0·32911	13
48	0·49206	0·47056	0·44942	0·42860	0·40809	0·38787	0·36793	0·34824	0·32878	12
49	0·49170	0·47021	0·44907	0·42826	0·40775	0·38754	0·36760	0·34791	0·32846	11
50	0·49134	0·46985	0·44872	0·42791	0·40741	0·38721	0·36727	0·34758	0·32814	10
51	0·49098	0·46950	0·44837	0·42757	0·40707	0·38687	0·36694	0·34726	0·32782	9
52	0·49062	0·46914	0·44802	0·42722	0·40674	0·38654	0·36661	0·34693	0·32749	8
53	0·49026	0·46879	0·44767	0·42688	0·40640	0·38620	0·36628	0·34661	0·32717	7
54	0·48989	0·46843	0·44732	0·42654	0·40606	0·38587	0·36595	0·34628	0·32685	6
55	0·48953	0·46808	0·44697	0·42619	0·40572	0·38553	0·36562	0·34596	0·32653	5
56	0·48917	0·46772	0·44662	0·42585	0·40538	0·38520	0·36529	0·34563	0·32621	4
57	0·48881	0·46737	0·44627	0·42551	0·40504	0·38487	0·36496	0·34530	0·32588	3
58	0·48845	0·46702	0·44593	0·42516	0·40470	0·38453	0·36463	0·34498	0·32556	2
59	0·48809	0·46666	0·44558	0·42482	0·40436	0·38420	0·36430	0·34465	0·32524	1
60	0·48773	0·46631	0·44523	0·42447	0·40403	0·38386	0·36397	0·34433	0·32492	0
′	26°	25°	24°	23°	22°	21°	20°	19°	18°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	72°	73°	74°	75°	76°	77°	78°	79°	80°	′
0	0·32492	0·30573	0·28675	0·26795	0·24933	0·23087	0·21256	0·19438	0·17633	60
1	0·32460	0·30541	0·28643	0·26764	0·24902	0·23056	0·21225	0·19408	0·17603	59
2	0·32428	0·30509	0·28612	0·26733	0·24871	0·23026	0·21195	0·19378	0·17573	58
3	0·32396	0·30478	0·28580	0·26701	0·24840	0·22995	0·21164	0·19347	0·17543	57
4	0·32363	0·30446	0·28549	0.26670	0·24809	0·22964	0·21134	0·19317	0·17513	56
5	0·32331	0·30414	0·28517	0·26639	0·24778	0·22934	0·21104	0·19287	0·17483	55
6	0·32299	0·30382	0·28486	0·26608	0·24747	0·22903	0·21073	0·19257	0·17453	54
7	0·32267	0·30351	0·28454	0·26577	0·24717	0·22872	0·21043	0·19227	0·17423	53
8	0·32235	0·30319	0·28423	0·26546	0·24686	0·22842	0·21013	0·19197	0·17393	52
9	0·32203	0·30287	0·28391	0·26515	0·24655	0·22811	0·20982	0·19166	0·17363	51
10	0·32171	0·30255	0·28360	0·26483	0·24624	0·22781	0·20952	0·19136	0·17333	50
11	0·32139	0·30224	0·28329	0·26452	0·24593	0·22750	0·20921	0·19106	0·17303	49
12	0·32106	0·30192	0·28297	0·26421	0·24562	0·22719	0·20891	0·19076	0·17273	48
13	0·32074	0·30160	0·28266	0·26390	0·24532	0·22689	0·20861	0·19046	0·17243	47
14	0·32042	0·30128	0·28234	0·26359	0·24501	0·22658	0·20830	0·19016	0·17213	46
15	0·32010	0·30097	0·28203	0·26328	0·24470	0·22628	0·20800	0·18986	0·17183	45
16	0·31978	0·30065	0·28172	0·26297	0·24439	0·22597	0·20770	0·18955	0·17153	44
17	0·31946	0·30033	0·28140	0·26266	0·24408	0·22567	0·20739	0·18925	0·17123	43
18	0·31914	0·30001	0·28109	0·26235	0·24377	0·22536	0·20709	0·18895	0·17093	42
19	0·31882	0·29970	0·28077	0·26203	0·24347	0·22505	0·20679	0·18865	0·17063	41
20	0·31850	0·29938	0·28046	0·26172	0·24316	0·22475	0·20648	0·18835	0·17033	40
21	0·31818	0·29906	0·28015	0·26141	0·24285	0·22444	0·20618	0·18805	0·17004	39
22	0·31786	0·29875	0·27983	0·26110	0·24254	0·22414	0·20588	0·18775	0·16974	38
23	0·31754	0·29843	0·27952	0·26079	0·24223	0·22383	0·20557	0·18745	0·16944	37
24	0·31722	0·29811	0·27921	0·26048	0·24193	0·22353	0·20527	0·18714	0·16914	36
25	0·31690	0·29780	0·27889	0·26017	0·24162	0·22322	0·20497	0·18684	0·16884	35
26	0·31658	0·29748	0·27858	0·25986	0·24131	0·22292	0·20466	0·18654	0·16854	34
27	0·31626	0·29716	0·27826	0·25955	0·24100	0·22261	0·20436	0·18624	0·16824	33
28	0·31594	0·29685	0·27795	0·25924	0·24069	0·22231	0·20406	0·18594	0·16794	32
29	0·31562	0·29653	0·27764	0·25893	0·24039	0·22200	0·20376	0·18564	0·16764	31
30	0·31530	0·29621	0·27732	0·25862	0·24008	0·22169	0·20345	0·18534	0·16734	30
31	0·31498	0·29590	0·27701	0·25831	0·23977	0·22139	0·20315	0·18504	0·16704	29
32	0·31466	0·29558	0·27670	0·25800	0·23946	0·22108	0·20285	0·18474	0·16674	28
33	0·31434	0·29526	0·27638	0·25769	0·23916	0·22078	0·20254	0·18444	0·16645	27
34	0·31402	0·29495	0·27607	0·25738	0·23885	0·22047	0·20224	0·18414	0·16615	26
35	0·31370	0·29463	0·27576	0·25707	0·23854	0·22017	0·20194	0·18384	0·16585	25
36	0·31338	0·29432	0·27545	0·25676	0·23823	0·21986	0·20164	0·18353	0·16555	24
37	0·31306	0·29400	0·27513	0·25645	0·23793	0·21956	0·20133	0·18323	0·16525	23
38	0·31274	0·29368	0·27482	0·25614	0·23762	0·21925	0·20103	0·18293	0·16495	22
39	0·31242	0·29337	0·27451	0·25583	0·23731	0·21895	0·20073	0·18263	0·16465	21
40	0·31210	0·29305	0·27419	0·25552	0·23700	0·21864	0·20042	0·18233	0·16435	20
41	0·31178	0·29274	0·27388	0·25521	0·23670	0·21834	0·20012	0·18203	0·16405	19
42	0·31147	0·29242	0·27357	0·25490	0·23639	0·21804	0·19982	0·18173	0·16376	18
43	0·31115	0·29210	0·27326	0·25459	0·23608	0·21773	0·19952	0·18143	0·16346	17
44	0·31083	0·29179	0·27294	0·25428	0·23578	0·21743	0·19921	0·18113	0·16316	16
45	0·31051	0·29147	0·27263	0·25397	0·23547	0·21712	0·19891	0·18083	0·16286	15
46	0·31019	0·29116	0·27232	0·25366	0·23516	0·21682	0·19861	0·18053	0·16256	14
47	0·30987	0·29084	0·27201	0·25335	0·23485	0·21651	0·19831	0·18023	0·16226	13
48	0·30955	0·29053	0·27169	0·25304	0·23455	0·21621	0·19801	0·17993	0·16196	12
49	0·30923	0·29021	0·27138	0·25273	0·23424	0·21590	0·19770	0·17963	0·16167	11
50	0·30891	0·28990	0·27107	0·25242	0·23393	0·21560	0·19740	0·17933	0·16137	10
51	0·30860	0·28958	0·27076	0·25211	0·23363	0·21529	0·19710	0·17903	0·16107	9
52	0·30828	0·28927	0·27044	0·25180	0·23332	0·21499	0·19680	0·17873	0·16077	8
53	0·30796	0·28895	0·27013	0·25149	0·23301	0·21469	0·19649	0·17843	0·16047	7
54	0·30764	0·28864	0·26982	0·25118	0·23271	0·21438	0·19619	0·17813	0·16017	6
55	0·30732	0·28832	0·26951	0·25087	0·23240	0·21408	0·19589	0·17783	0·15988	5
56	0·30700	0·28801	0·26920	0·25056	0·23209	0·21377	0·19559	0·17753	0·15958	4
57	0·30669	0·28769	0·26888	0·25026	0·23179	0·21347	0·19529	0·17723	0·15928	3
58	0·30637	0·28738	0·26857	0·24995	0·23148	0·21316	0·19498	0·17693	0·15898	2
59	0·30605	0·28706	0·26826	0·24964	0·23117	0·21286	0·19468	0·17663	0·15868	1
60	0·30573	0·28675	0·26795	0·24933	0·23087	0·21256	0·19438	0·17633	0·15838	0
′	17°	16°	15°	14°	13°	12°	11°	10°	9°	′

NATURAL TANGENTS.

NATURAL COTANGENTS.

′	81°	82°	83°	84°	85°	86°	87°	88°	89°	′
0	0·15838	0·14054	0·12278	0·10510	0·08749	0·06993	0·05241	0·03492	0·01746	60
1	0·15809	0·14024	0·12249	0·10481	0·08720	0·06963	0·05212	0·03463	0·01716	59
2	0·15779	0·13995	0·12219	0·10452	0·08690	0·06934	0·05182	0·03434	0·01687	58
3	0·15749	0·13965	0·12190	0·10422	0·08661	0·06905	0·05153	0·03405	0·01658	57
4	0·15719	0·13935	0·12160	0·10393	0·08632	0·06876	0·05124	0·03376	0·01629	56
5	0·15689	0·13906	0·12131	0·10363	0·08602	0·06847	0·05095	0·03346	0·01600	55
6	0·15660	0·13876	0·12101	0·10334	0·08573	0·06817	0·05066	0·03317	0·01571	54
7	0·15630	0·13846	0·12072	0·10305	0·08544	0·06788	0·05037	0·03288	0·01542	53
8	0·15600	0·13817	0·12042	0·10275	0·08514	0·06759	0·05007	0·03259	0·01513	52
9	0·15570	0·13787	0·12013	0·10246	0·08485	0·06730	0·04978	0·03230	0·01484	51
10	0·15540	0·13758	0·11983	0·10216	0·08456	0·06700	0·04949	0·03201	0·01455	50
11	0·15511	0·13728	0·11954	0·10187	0·08427	0·06671	0·04920	0·03172	0·01425	49
12	0·15481	0·13698	0·11924	0·10158	0·08397	0·06642	0·04891	0·03143	0·01396	48
13	0·15451	0·13669	0·11895	0·10128	0·08368	0·06613	0·04862	0·03114	0·01367	47
14	0·15421	0·13639	0·11865	0·10099	0·08339	0·06584	0·04833	0·03084	0·01338	46
15	0·15391	0·13609	0·11836	0·10069	0·08309	0·06554	0·04803	0·03055	0·01309	45
16	0·15362	0·13580	0·11806	0·10040	0·08280	0·06525	0·04774	0·03026	0·01280	44
17	0·15332	0·13550	0·11777	0·10011	0·08251	0·06496	0·04745	0·02997	0·01251	43
18	0·15302	0·13521	0·11747	0·09981	0·08221	0·06467	0·04716	0·02968	0·01222	42
19	0·15272	0·13491	0·11718	0·09952	0·08192	0·06438	0·04687	0·02939	0·01193	41
20	0·15243	0·13461	0·11688	0·09923	0·08163	0·06408	0·04658	0·02910	0·01164	40
21	0·15213	0·13432	0·11659	0·09893	0·08134	0·06379	0·04628	0·02881	0·01135	39
22	0·15183	0·13402	0·11629	0·09864	0·08104	0·06350	0·04599	0·02851	0·01105	38
23	0·15153	0·13372	0·11600	0·09834	0·08075	0·06321	0·04570	0·02822	0·01076	37
24	0·15124	0·13343	0·11570	0·09805	0·08046	0·06291	0·04541	0·02793	0·01047	36
25	0·15094	0·13313	0·11541	0·09776	0·08017	0·06262	0·04512	0·02764	0·01018	35
26	0·15064	0·13284	0·11511	0·09746	0·07987	0·06233	0·04483	0·02735	0·00989	34
27	0·15034	0·13254	0·11482	0·09717	0·07958	0·06204	0·04454	0·02706	0·00960	33
28	0·15005	0·13224	0·11452	0·09688	0·07929	0·06175	0·04424	0·02677	0·00931	32
29	0·14975	0·13195	0·11423	0·09658	0·07899	0·06145	0·04395	0·02648	0·00902	31
30	0·14945	0·13165	0·11394	0·09629	0·07870	0·06116	0·04366	0·02619	0·00873	30
31	0·14915	0·13136	0·11364	0·09600	0·07841	0·06087	0·04337	0·02589	0·00844	29
32	0·14886	0·13106	0·11335	0·09570	0·07812	0·06058	0·04308	0·02560	0·00815	28
33	0·14856	0·13076	0·11305	0·09541	0·07782	0·06029	0·04279	0·02531	0·00785	27
34	0·14826	0·13047	0·11276	0·09511	0·07753	0·05999	0·04250	0·02502	0·00756	26
35	0·14796	0·13017	0·11246	0·09482	0·07724	0·05970	0·04220	0·02473	0·00727	25
36	0·14767	0·12988	0·11217	0·09453	0·07695	0·05941	0·04191	0·02444	0·00698	24
37	0·14737	0·12958	0·11187	0·09423	0·07665	0·05912	0·04162	0·02415	0·00669	23
38	0·14707	0·12929	0·11158	0·09394	0·07636	0·05883	0·04133	0·02386	0·00640	22
39	0·14678	0·12899	0·11128	0·09365	0·07607	0·05854	0·04104	0·02357	0·00611	21
40	0·14648	0·12869	0·11099	0·09335	0·07578	0·05824	0·04075	0·02328	0·00582	20
41	0·14618	0·12840	0·11070	0·09306	0·07548	0·05795	0·04046	0·02298	0·00553	19
42	0·14588	0·12810	0·11040	0·09277	0·07519	0·05766	0·04016	0·02269	0·00524	18
43	0·14559	0·12781	0·11011	0·09247	0·07490	0·05737	0·03987	0·02240	0·00495	17
44	0·14529	0·12751	0·10981	0·09218	0·07461	0·05708	0·03958	0·02211	0·00465	16
45	0·14499	0·12722	0·10952	0·09189	0·07431	0·05678	0·03929	0·02182	0·00436	15
46	0·14470	0·12692	0·10922	0·09159	0·07402	0·05649	0·03900	0·02153	0·00407	14
47	0·14440	0·12662	0·10893	0·09130	0·07373	0·05620	0·03871	0·02124	0·00378	13
48	0·14410	0·12633	0·10863	0·09101	0·07344	0·05591	0·03842	0·02095	0·00349	12
49	0·14381	0·12603	0·10834	0·09071	0·07314	0·05562	0·03812	0·02066	0·00320	11
50	0·14351	0·12574	0·10805	0·09042	0·07285	0·05533	0·03783	0·02036	0·00291	10
51	0·14321	0·12544	0·10775	0·09013	0·07256	0·05503	0·03754	0·02007	0·00262	9
52	0·14291	0·12515	0·10746	0·08983	0·07227	0·05474	0·03725	0·01978	0·00233	8
53	0·14262	0·12485	0·10716	0·08954	0·07197	0·05445	0·03696	0·01949	0·00204	7
54	0·14232	0·12456	0·10687	0·08925	0·07168	0·05416	0·03667	0·01920	0·00175	6
55	0·14202	0·12426	0·10657	0·08895	0·07139	0·05387	0·03638	0·01891	0·00145	5
56	0·14173	0·12397	0·10628	0·08866	0·07110	0·05357	0·03609	0·01862	0·00116	4
57	0·14143	0·12367	0·10599	0·08837	0·07080	0·05328	0·03579	0·01833	0·00087	3
58	0·14113	0·12338	0·10569	0·08807	0·07051	0·05299	0·03550	0·01804	0·00058	2
59	0·14084	0·12308	0·10540	0·08778	0·07022	0·05270	0·03521	0·01775	0·00029	1
60	0·14054	0·12278	0·10510	0·08749	0·06993	0·05241	0·03492	0·01746	0·00000	0
′	8°	7°	6°	5°	4°	3°	2°	1°	0°	′

NATURAL TANGENTS.

APPENDIX II.

SOME USEFUL FORMULAE.

I. TRIGONOMETRICAL FUNCTIONS.

(1) *Definition.*—In a plane right-angled triangle $A\ B\ C$, in which $C = 90°$ and c the hypotenuse (*cf.* Fig. 82): $\sin A = a/c$; $\cos A =$

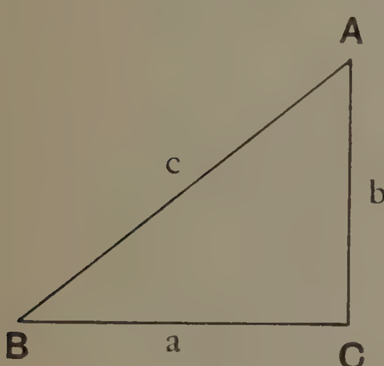

FIG. 82.

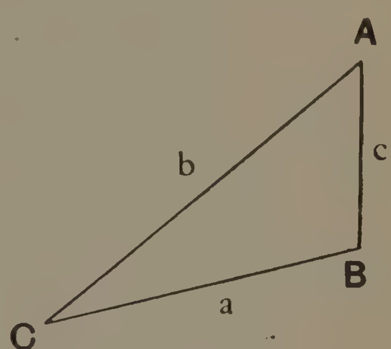

FIG. 83.

b/c; $\tan A = a/b$; $\cot A = b/a$; [$\operatorname{cosec} A = 1/\sin A$; $\sec A = 1/\cos A$]; $\tan A = \sin A/\cos A$; $\cot A = \cos A/\sin A$.

(2) *Complementary Values*... $\sin A = \cos (90° - A)$; $\cos A = \sin (90° - A)$; $\tan A = \cot (90° - A)$; $\cot A = \tan (90° - A)$.

(3) *Supplementary Values*... $\sin A = \sin (180° - A)$; $\cos A = -\cos (180° - A)$; $\tan A = -\tan (180° - A)$; $\cot A = -\cot (180° - A)$.

(4) *Various Formulae*... $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$;

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}; \quad \cot (A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A};$$

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B);$$

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B);$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B);$$

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B);$$

$$\frac{1 + \tan A}{1 - \tan A} = \tan (45° + A); \quad \frac{\cot A + 1}{\cot A - 1} = \cot (45° - A).$$

II. SOLUTION OF PLANE TRIANGLES.

(1) In an Oblique Triangle A, B, C, with opposite sides a, b, c (*cf.* Fig. 83), $a/\sin A = b/\sin B = c/\sin C$; whence, $a/c = \sin A/\sin C$.

(2) **Given a, c and B.** Then, $\tan \frac{1}{2}(A - C) = \frac{a-c}{a+c} \cdot \cot \frac{1}{2}B$; whence $\frac{1}{2}(A - C)$ is evaluated. Further, as $\frac{1}{2}(A + C) = 90° - \frac{1}{2}B$, we obtain A by addition and C by subtraction. Finally, $b = c \sin B/\sin C = a \sin B/\sin A$.

III. SOLUTION OF SPHERICAL TRIANGLES.

A. RIGHT-ANGLED TRIANGLES.

The various formulae for the solution of a right-angled triangle can be simply epitomised in the form of Napier's rules. The five non-rectangular angles of the triangle (*cf.* Fig. 84) are arranged in a regular

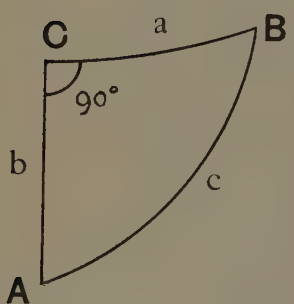

FIG. 84.

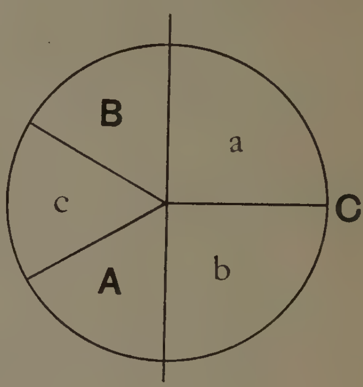

FIG. 85.

order in the five-sector arrangement of Fig. 85, care being taken to place the two sides, including the right angle, in the right-hand pair of compartments. Then:

sine of any angle = product of 'opposite' cosines,
= product of 'adjacent' tangents—

provided that *complementary* values are always taken in the case of the three angles A, c, B, which lie in the left half of the diagram.

Note.—The angles B and b, for example, are 'adjacent' to a. Therefore, according to the second rule, we have:

$$\sin a = \tan b \cdot \tan (90° - B) = \tan b \cdot \cot B.$$

Again, the angles a and b are 'opposite' to c. Therefore, according to the first rule, we have:

$$\sin (90° - c) = \cos a \cdot \cos b; \ \textit{i.e.} \ \cos c = \cos a \cdot \cos b.$$

B. OBLIQUE-ANGLED TRIANGLES.

Preliminary Notes.—It has already been pointed out in connexion with the cotangent form of the Millerian zonal formula (p. 62) that a computed angle close to 90° is liable to be disproportionately removed from the true value; and that any measured value close to 90° should therefore be selected as a fundamental. Exactly similar considerations hold for the sine of an angle close to 90°. No reliance, for example, can be placed on an angle α, β or γ of the orthodox form of elements (or of an angle e or g of the angular form) which is close to 90° and has been computed from a formula of the type: $\sin B = \sin b \,.\, \sin A/\sin a$; and any such elemental angle should obviously be measured, if a two-circle instrument be available, or alternatively be computed by such formulæ as those given below under '*Case* 1.'

With the exception of the case in which all three sides (or all three angles) are given, an oblique triangle may be solved in terms of two component right-angled

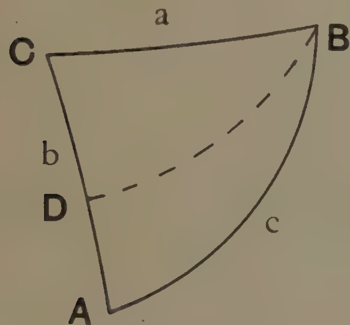

FIG. 86.

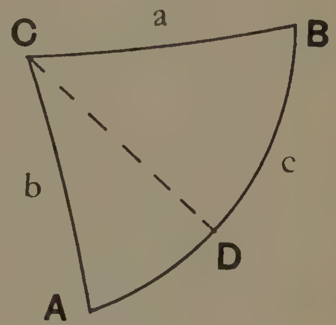

FIG. 87.

triangles, as indicated by Figs. 86-87 (for the instrumental resolution of an oblique spherical triangle *cf.* Fig. 20, p. 17). Thus, if the two sides a, b and the included angle C should be given, and the arc BD be imagined as drawn normal to CA, then CD and DB (as also CBD) can be computed from the data a and C; and, as DA (equal to $CA—CD$) and DB are both now known, it is possible to evaluate the second right-angled triangle. Fig. 87 represents a case in which A, B and a might be the original data.

In order to detect mistakes as soon as possible, and so avoid consequential errors, it is advisable to check immediately in projection every computed value by the part 'C' of the crystallographic protractor. This procedure also serves to remove any ambiguity arising from the fact that the sine of an angle has the same value as the sine of its supplement.

Case 1. **Given the three sides a, b, c.**—Putting *half* the sum of the three sides equal to s, we have:

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \,.\, \sin c}}; \text{ or } \cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \,.\, \sin(s-a)}{\sin b \,.\, \sin c}}.$$

Case 2. **Given the three angles A, B, C.**—Putting *half* the sum of the angles equal to S, we have:

$$\sin \tfrac{1}{2}a = \sqrt{\frac{\cos(180^\circ - S) \,.\, \cos(S-A)}{\sin B \,.\, \sin C}}; \text{ or,}$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos(S-B) \,.\, \cos(S-C)}{\sin B \,.\, \sin C}}.$$

Case 3.—**Given Two Sides, a and b, and the Included Angle, C.**—The angles A and B are found simultaneously by the Napierian formulae:

$$\left.\begin{aligned}\tan\tfrac{1}{2}(A+B)&=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cdot\cot\tfrac{1}{2}C\\ \tan\tfrac{1}{2}(A-B)&=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cdot\cot\tfrac{1}{2}C\end{aligned}\right\}$$

Whence we obtain the value of A by addition, and the value of B by subtraction.

The value of c can then be obtainable by one of the formulae:

$$\sin c\ /\ \sin C=\sin a\ /\ \sin A=\sin b\ /\ \sin B.$$

Note.—It is sometimes c that is wanted, and not A and B, in which case its value can be computed, either by the formula:

$$\cos c=\cos a\,.\cos b+\sin a\,.\sin b\,.\cos C,$$

or in two purely logarithmic stages by the following pair of formulae:

$$\tan x=\tan a\,.\cos C;$$

$$\cos c=\frac{\cos a\,.\cos(b-x)}{\cos x}.$$

Case 4. **Given Two Angles, A and B, and the Included Side, c.**—The sides a and b are found simultaneously by the Napierian formulae:

$$\left.\begin{aligned}\tan\tfrac{1}{2}(a+b)&=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\cdot\tan\tfrac{1}{2}c\\ \tan\tfrac{1}{2}(a-b)&=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\cdot\tan\tfrac{1}{2}c\end{aligned}\right\}$$

Whence we obtain the value of a by addition, and the value of b by subtraction.

The value of C is then obtainable by one of the formulae:

$$\sin C\ /\ \sin c=\sin B\ /\ \sin b=\sin A\ /\ \sin a.$$

Note.—Ambiguity as to the value of C is generally removable, as in the preceding case, by a graphical determination. Wherever C lies between the limits $89°—91°$, the value should be checked by the use of the Delambre-Gauss formula:

$$\cos\tfrac{1}{2}C=\sin\tfrac{1}{2}(A+B)\,.\cos\tfrac{1}{2}c\ /\ \cos\tfrac{1}{2}(a-b);$$

or, independently of a and b, by the two formulae:

$$\cot x=\tan B\cos c;$$

$$\cos C=\frac{\cos B\sin(A-x)}{\sin x}.$$

Case 5. **Given Two Sides, a and b, and an Angle, A, opposite to one of them.**—This case may be solved in two stages:

(1) The value of B is obtained from the formula, $\sin B\ /\ \sin b=\sin A\ /\ \sin a$.

(2) The values of C and c are then found by the Napierian formulae:

$$\tan\tfrac{1}{2}C=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cdot\cot\tfrac{1}{2}(A+B);$$

$$\tan\tfrac{1}{2}c=\frac{\cos\frac{1}{2}(A+B}{\cos\frac{1}{2}(A-B)}\cdot\tan\tfrac{1}{2}(a+b).$$

Case 6. **Given two Angles, A and B, and the Side, a, opposite one of them.**—The value of b is obtained from the formula $\sin b / \sin B = \sin a / \sin A$. The subsequent procedure being exactly as under Case 5 above.

IV. TAUTOZONAL AND COPLANAR QUARTETTES.

A. GENERAL (NON-RECTANGULAR) CASE.

The general Millerian formulae hold equally for the three cases respectively presented by Figs. 88—90, in which Fig. 89 refers to the ϕ-angles of 2-circle goniometry (zone adjustment).

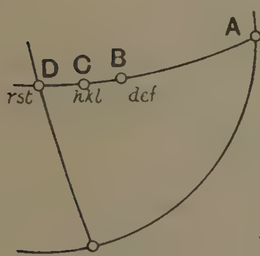

FIG. 88.

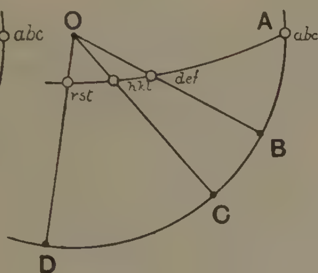

FIG. 89.

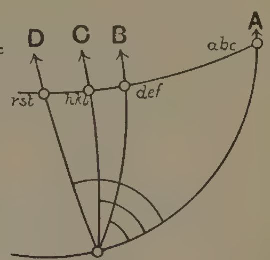

FIG. 90.

Case 1.—**Given all the angles and three sets of indices.**

$$\frac{\sin AB}{\sin AC}\cdot\frac{\sin DC}{\sin DB}=\frac{\dfrac{abc}{def}}{\dfrac{abc}{hkl}}\cdot\frac{\dfrac{rst}{hkl}}{\dfrac{rst}{def}}$$

$$\text{i.e. } \frac{ae-db}{ak-hb}\cdot\frac{rk-hs}{re-ds} \text{ or } \frac{af-dc}{al-hc}\cdot\frac{rl-ht}{rf-dt} \text{ or } \frac{bf-ec}{bl-kc}\cdot\frac{sl-kt}{sf-et}.$$

Case 2. **Given all the indices and two angles.**

$$\frac{\cot AC-\cot AD}{\cot AB-\cot AD}=\frac{\dfrac{abc}{def}}{\dfrac{abc}{hkl}}\cdot\frac{\dfrac{rst}{hkl}}{\dfrac{rst}{def}}=\frac{p}{q}; \text{ whence}$$

$$p\cot AB - q\cot AC = (p-q)\cot AD.$$

Note.—In the special 'harmonic case' (see p. 63) the equation takes the form:

$$\cot AB + \cot AD = 2\cot AC.$$

B. SPECIAL (RECTANGULAR) CASE. Consult 'Multiple tangent table.'

The special Millerian formulae hold equally for the three cases respectively presented by Figs. 91—93, in which Fig. 92 refers to the ϕ-angles of 2-circle goniometry (zone adjustment).

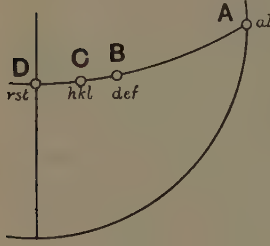

FIG. 91.

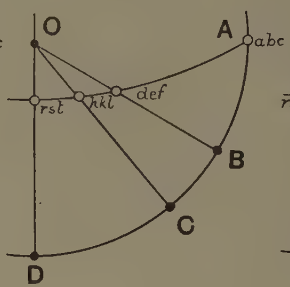

FIG. 92.

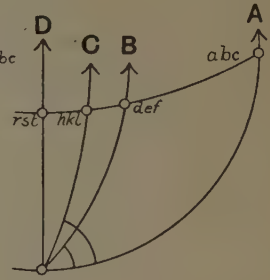

FIG. 93.

Case 1. **Given all the angles and three sets of indices.**

$$\frac{\dfrac{abc}{def}}{\dfrac{abc}{hkl}} \cdot \frac{\dfrac{rst}{hkl}}{\dfrac{rst}{def}} = \frac{\tan AB}{\tan AC}.$$

Case 2. **Given all the indices and one angle (either AB or AC).**

$$\frac{\tan AB}{\tan AC} = \frac{\dfrac{abc}{def}}{\dfrac{abc}{hkl}} \cdot \frac{\dfrac{rst}{hkl}}{\dfrac{rst}{def}} = \frac{p}{q}; \text{ whence } \tan AB = (p/q) \tan AC.$$

V. TWO-CIRCLE TRANSFORMATION FORMULA.

The following formula giving the value of an interfacial angle PQ in terms of the two-circle readings P $(\phi_1\ \rho_1)$ and Q $(\phi_2\ \rho_2)$ will be of use to a single-circle worker who has occasion to re-measure a crystal which has been previously described by a two-circle worker. The formula is quite general, *i.e.*, independent of the type of adjustment :

$$\cos PQ = \cos \rho_1 \,.\, \cos \rho_2 + \sin \rho_1 \,.\, \sin \rho_2 \,.\, \cos (\phi_2 - \phi_1)$$

Note.—The 'transformation formula' is nothing more nor less than the special formula (given on p. 132 under *Case* 3) allowing the computation of the 'base' of an oblique spherical triangle from the other two sides and the apical angle. The formula can accordingly be transformed into two purely logarithmic components :

$$\tan x = \tan \rho_1 \,.\, \cos (\phi_2 - \phi_1)\,;$$

$$\cos PQ = \frac{\cos \rho_1 \cdot \cos (\rho_2 - x)}{\cos x}.$$

VI. CRYSTAL ELEMENTS.

A. FEDOROV'S ANGULAR ELEMENTS.

Anorthic System.—The angular elements are the angles d, e, ab, f, g, of Fig. 94, in which ab is an interfacial angle; e is equal to $(180°—\alpha)$, and g is equal to $(180°—\beta)$.

[In the monoclinic system they acquire the special form d, f, g.]

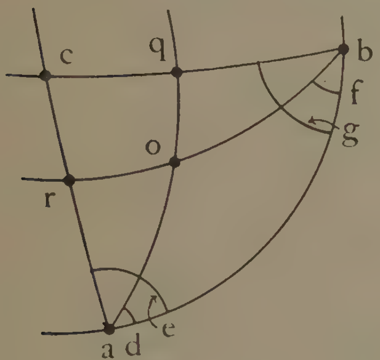

FIG. 94.

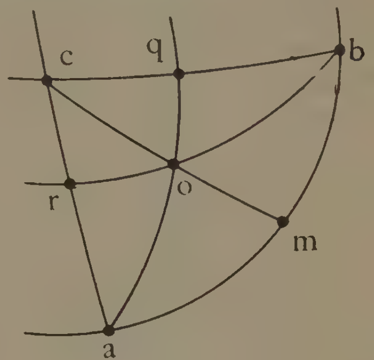

FIG. 95.

B. ORTHODOX ELEMENTS.

Case 1. **Anorthic System.**—The elements are $a : b : c$, in which b is taken as unity, α, β and γ (the latter being the supplements of the angles *cab*, *abc* and *bca* respectively, of Fig. 95).

$$a : b = \sin acm \,/\, \sin bcm ; \; c : b = \sin caq \,/\, \sin baq.$$
$$[c : a = \sin cbr/\sin abr].$$

Case 2. **Monoclinic System.**—The elements are $a : b : c$, β (*cf.* Fig. 96), in which $\beta = 180° - abc$).

$$a : b = \tan acm ; \;\; c : b = \tan caq.$$
$$[c : a = \sin cbr/\sin abr].$$

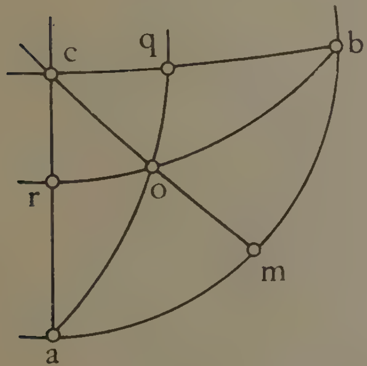

FIG. 96.

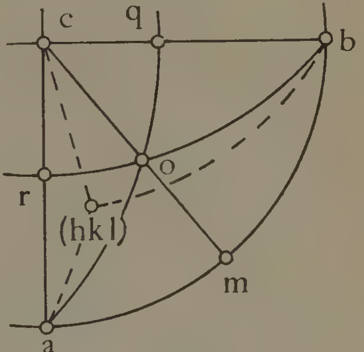

FIG. 97.

Case 3. **Orthorhombic System.**—The elements are $a : b : c$ (*cf.* Fig. 97).

$a : b = \tan am(100 : 110) = (h/k) \tan 100 : hk0.$

$c : b = \tan cq(001 : 011) = (l/k) \tan 001 : 0kl.$

$(a/h) \cos 100 : hkl = (b/k) \cos 010 : hkl = (c/l) \cos 001 : hkl.$

Case 4. **Tetragonal System.**—The element is $c : a$.

$c : a = \tan 001 : 101 = \cos 45° . \tan 001 : 111$ [$L\cos 45° = 9{\cdot}84949$].

Case 5. **Hexagonal System.**—The element is $c : a$ (*cf.* Fig. 98).

$c : a = \tan(0001 : 10\bar{1}1) \cos 30°$ [$L \cos 30° = 9{\cdot}93753$] $= \tan 0001 : 11\bar{2}2.$

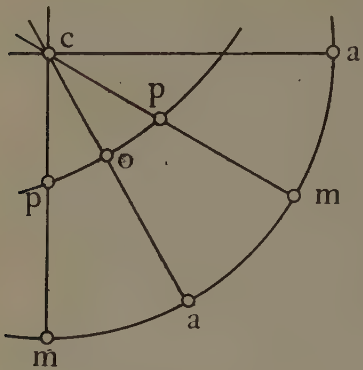

FIG. 98.

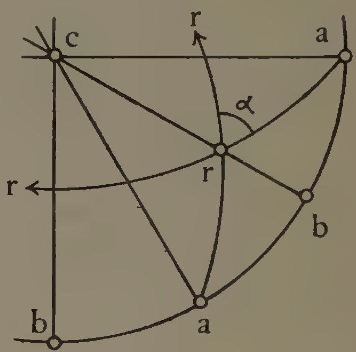

FIG. 99.

Case 6. **Rhombohedral System.**—The element is the angle marked α in Fig. 99; the angle *cr* would be much more useful.

$$\cos \tfrac{1}{2}\alpha = \cos 60° / \sin \tfrac{1}{2}rr \quad [L \cos 60° = 9{\cdot}69897].$$

APPENDIX III.

MULTIPLE TANGENT TABLES.

Preliminary Notes.—As explained in the text, such tables are only applicable to the special case of the rectangular zone, and afford a permanent solution of the equation, $\tan AB = (p/q) \tan AC$, in two senses, one the converse of the other:

(1) The first sense is that in which *both* angular values AB and AC (say, their *measured* values) are given, and it is desired to deduce the value p/q as being a clue to the indices, either of B or of C in terms of the other. **In this case the 'supplementary table' of p. 149 should be employed,** because whilst satisfying every reasonable demand for accuracy it is the more compact of the two tables. As explained on p. 83 it is presupposed that all angular values less than 75° are rounded off to the nearest $\frac{1}{2}$°, and those above 15° to the nearest $\frac{1}{4}$°.

(2) The second sense is that in which all the indices (and therefore p/q) are known, as well as *one* of the two angular values, AB or AC; and it is desired to derive the theoretical value of the *other* to the nearest minute. In this case the main table (pp. 137—148) must naturally be employed (as explained on p. 70). Apart from its more obvious field of application, this main table can be invoked for the purpose of a mutual adjustment of a sequence of faces (in a rectangular zone) with a view to the attainment of elements of enhanced accuracy (*cf.* p. 100). But as it is scarcely desirable to improve on the high standard of accuracy, established by Wollaston in 1812 and conserved by successive generations of crystallographers, the table can be more profitably applied to a reduction of the number of crystals measured. A single orthorhombic (or monoclinic) crystal, for example, exhibiting n prism forms ($hk0$) can be thus made to yield a ratio $a : b$, equal in accuracy to that demanded by n crystals treated on classical lines.

1 (5°00—11°00) 2 (9°55—21°15) 3 (14°42—30°15) 4 (19°17—37°52) 5 (23°38—44°11)

1	2	3	4	5	1	2	3	4	5
5°00	9°55	14°42	19°17	23°38	8°00	15°42	22°52	29°21	35°06
5°03	10°01	14°51	19°28	23°50	8°03	15°48	22°59	29°30	35°16
5°06	10°07	14°59	19°39	24°03	8°06	15°53	23°07	29°39	35°26
5°09	10°13	15°08	19°50	24°16	8°09	15°59	23°15	29°48	35°36
5°12	10°19	15°16	20°00	24°28	8°12	16°05	23°23	29°58	35°46
5°15	10°25	15°25	20°11	24°41	8°15	16°10	23°30	30°07	35°56
5°18	10°31	15°33	20°22	24°53	8°18	16°16	23°38	30°16	36°06
5°21	10°37	15°42	20°32	25°05	8°21	16°22	23°46	30°25	36°16
5°24	10°42	15°50	20°43	25°18	8°24	16°27	23°54	30°34	36°26
5°27	10°48	15°58	20°53	25°30	8°27	16°33	24°01	30°43	36°36
5°30	10°54	16°07	21°04	25°42	8°30	16°38	24°09	30°52	36°46
5°33	11°00	16°15	21°14	25°55	8°33	16°44	24°17	31°01	36°56
5°36	11°06	16°23	21°25	26°07	8°36	16°50	24°24	31°10	37°06
5°39	11°12	16°32	21°35	26°19	8°39	16°55	24°32	31°19	37°15
5°42	11°17	16°40	21°46	26°31	8°42	17°01	24°39	31°28	37°25
5°45	11°23	16°48	21°56	26°43	8°45	17°07	24°47	31°37	37°35
5°48	11°29	16°57	22°07	26°56	8°48	17°12	24°55	31°46	37°44
5°51	11°35	17°05	22°17	27°08	8°51	17°18	25°02	31°55	37°54
5°54	11°41	17°13	22°28	27°20	8°54	17°23	25°10	32°04	38°04
5°57	11°46	17°22	22°38	27°31	8°57	17°29	25°17	32°13	38°13
6°00	11°52	17°30	22°48	27°43	9°00	17°35	25°25	32°21	38°23
6°03	11°58	17°38	22°59	27°55	9°03	17°40	25°32	32°30	38°32
6°06	12°04	17°47	23°09	28°07	9°06	17°46	25°39	32°39	38°41
6°09	12°10	17°55	23°19	28°19	9°09	17°51	25°47	32°48	38°51
6°12	12°15	18°03	23°29	28°31	9°12	17°57	25°55	32°56	39°00
6°15	12°21	18°11	23°39	28°42	9°15	18°02	26°02	33°05	39°09
6°18	12°27	18°20	23°50	28°54	9°18	18°08	26°10	33°14	39°19
6°21	12°33	18°28	24°00	29°06	9°21	18°14	26°17	33°22	39°28
6°24	12°39	18°36	24°10	29°17	9°24	18°19	26°25	33°31	39°37
6°27	12°44	18°44	24°20	29°29	9°27	18°25	26°32	33°39	39°46
6°30	12°50	18°52	24°30	29°40	9°30	18°30	26°39	33°48	39°55
6°33	12°56	19°00	24°40	29°52	9°33	18°36	26°47	33°56	40°04
6°36	13°02	19°09	24°50	30°03	9°36	18°41	26°54	34°05	40°13
6°39	13°08	19°17	25°00	30°14	9°39	18°47	27°02	34°13	40°22
6°42	13°13	19°25	25°10	30°26	9°42	18°52	27°09	34°22	40°31
6°45	13°19	19°33	25°20	30°37	9°45	18°58	27°16	34°30	40°40
6°48	13°25	19°41	25°30	30°48	9°48	19°03	27°24	34°38	40°49
6°51	13°31	19°49	25°40	30°59	9°51	19°09	27°31	34°47	40°58
6°54	13°36	19°57	25°50	31°11	9°54	19°15	27°38	34°55	41°07
6°57	13°42	20°05	26°00	31°22	9°57	19°20	27°45	35°03	41°15
7°00	13°48	20°13	26°09	31°33	10°00	19°26	27°53	35°12	41°24
7°03	13°54	20°21	26°19	31°44	10°03	19°31	28°00	35°20	41°33
7°06	13°59	20°29	26°29	31°55	10°06	19°37	28°07	35°28	41°41
7°09	14°05	20°37	26°39	32°06	10°09	19°42	28°14	35°36	41°50
7°12	14°11	20°45	26°49	32°17	10°12	19°48	28°22	35°45	41°59
7°15	14°17	20°53	26°58	32°28	10°15	19°53	28°29	35°53	42°07
7°18	14°22	21°01	27°08	32°38	10°18	19°58	28°36	36°01	42°15
7°21	14°28	21°09	27°18	32°49	10°21	20°04	28°43	36°09	42°24
7°24	14°34	21°17	27°27	33°00	10°24	20°09	28°50	36°17	42°32
7°27	14°39	21°25	27°37	33°11	10°27	20°15	28°57	36°25	42°41
7°30	14°45	21°33	27°46	33°21	10°30	20°20	29°04	36°33	42°49
7°33	14°51	21°41	27°56	33°32	10°33	20°26	29°12	36°41	42°58
7°36	14°57	21°49	28°05	33°43	10°36	20°31	29°19	36°49	43°06
7°39	15°02	21°57	28°15	33°53	10°39	20°37	29°26	36°57	43°14
7°42	15°08	22°05	28°24	34°04	10°42	20°42	29°33	37°05	43°22
7°45	15°14	22°13	28°34	34°14	10°45	20°48	29°40	37°13	43°31
7°48	15°19	22°20	28°43	34°24	10°48	20°53	29°47	37°21	43°39
7°51	15°25	22°28	28°53	34°35	10°51	20°58	29°54	37°29	43°47
7°54	15°31	22°36	29°02	34°45	10°54	21°04	30°01	37°36	43°55
7°57	15°36	22°44	29°11	34°55	10°57	21°09	30°08	37°44	44°03
8°00	15°42	22°52	29°21	35°06	11°00	21°15	30°15	37°52	44°11

1 (11°00—17°00) 2 (21°15—31°27) 3 (30°15—42°32) 4 (37°52—50°44) 5 (44°11—56°48)

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11°03	21°20	30°22	38°00	44°19	14°03	26°35	36°54	45°02	51°22
11°06	21°25	30°29	38°07	44°27	14°06	26°40	37°00	45°08	51°28
11°09	21°31	30°36	38°15	44°35	14°09	26°45	37°06	45°14	51°34
11°12	21°36	30°43	38°23	44°43	14°12	26°51	37°12	45°21	51°41
11°15	21°42	30°50	38°30	44°51	14°15	26°56	37°18	45°27	51°47
11°18	21°47	30°56	38°38	44°58	14°18	27°01	37°24	45°33	51°53
11°21	21°52	31°03	38°46	45°06	14°21	27°06	37°30	45°40	51°59
11°24	21°58	31°10	38°53	45°14	14°24	27°11	37°36	45°46	52°05
11°27	22°03	31°17	39°01	45°22	14°27	27°16	37°42	45°52	52°11
11°30	22°08	31°24	39°08	45°29	14°30	27°21	37°48	45°58	52°17
11°33	22°14	31°31	39°16	45°37	14°33	27°26	37°54	46°04	52°23
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11°39	22°25	31°44	39°31	45°52	14°39	27°36	38°06	46°17	52°35
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11°45	22°35	31°58	39°46	46°07	14°45	27°46	38°18	46°29	52°47
11°48	22°41	32°05	39°53	46°15	14°48	27°51	38°24	46°35	52°53
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11°54	22°51	32°18	40°08	46°30	14°54	28°01	38°36	46°47	53°04
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12°03	23°07	32°38	40°30	46°52	15°03	28°16	38°53	47°05	53°21
12°06	23°12	32°45	40°37	46°59	15°06	28°21	38°59	47°11	53°27
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12°15	23°28	33°05	40°58	47°21	15°15	28°36	39°17	47°29	53°44
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12°24	23°44	33°24	41°20	47°42	15°24	28°51	39°34	47°46	54°01
12°27	23°49	33°31	41°27	47°50	15°27	28°56	39°40	47°52	54°07
12°30	23°55	33°38	41°34	47°57	15°30	29°01	39°46	47°58	54°12
12°33	24°00	33°44	41°41	48°04	15°33	29°06	39°51	48°04	54°18
12°36	24°05	33°51	41°48	48°11	15°36	29°11	39°57	48°10	54°24
12°39	24°10	33°57	41°55	48°18	15°39	29°16	40°03	48°15	54°29
12°42	24°16	34°04	42°02	48°25	15°42	29°21	40°08	48°21	54°34
12°45	24°21	34°10	42°09	48°32	15°45	29°26	40°14	48°27	54°39
12°48	24°26	34°17	42°16	48°39	15°48	29°30	40°20	48°32	54°45
12°51	24°31	34°23	42°23	48°45	15°51	29°35	40°25	48°38	54°50
12°54	24°37	34°30	42°30	48°52	15°54	29°40	40°31	48°44	54°56
12°57	24°42	34°36	42°36	48°59	15°57	29°45	40°37	48°49	55°01
13°00	24°47	34°42	42°43	49°06	16°00	29°50	40°42	48°55	55°06
13°03	24°52	34°49	42°50	49°13	16°03	29°55	40°48	49°01	55°12
13°06	24°57	34°55	42°57	49°19	16°06	30°00	40°53	49°06	55°17
13°09	25°03	35°02	43°04	49°26	16°09	30°05	40°59	49°12	55°22
13°12	25°08	35°08	43°10	49°33	16°12	30°10	41°04	49°17	55°27
13°15	25°13	35°14	43°17	49°39	16°15	30°14	41°10	49°23	55°33
13°18	25°18	35°21	43°24	49°46	16°18	30°19	41°16	49°28	55°38
13°21	25°23	35°27	43°30	49°53	16°21	30°24	41°21	49°34	55°43
13°24	25°28	35°33	43°37	49°59	16°24	30°29	41°27	49°39	55°48
13°27	25°34	35°40	43°44	50°06	16°27	30°34	41°32	49°45	55°53
13°30	25°39	35°46	43°50	50°12	16°30	30°39	41°38	49°50	55°58
13°33	25°44	35°52	43°57	50°19	16°33	30°43	41°43	49°56	56°03
13°36	25°49	35°58	44°04	50°25	16°36	30°48	41°48	50°01	56°09
13°39	25°54	36°05	44°10	50°32	16°39	30°53	41°54	50°06	56°14
13°42	25°59	36°11	44°17	50°38	16°42	30°58	41°59	50°12	56°19
13°45	26°05	36°17	44°23	50°44	16°45	31°03	42°04	50°17	56°24
13°48	26°10	36°23	44°30	50°51	16°48	31°08	42°10	50°22	56°29
13°51	26°15	36°29	44°36	50°57	16°51	31°12	42°16	50°28	56°34
13°54	26°20	36°35	44°43	51°03	16°54	31°17	42°21	50°33	56°39
13°57	26°25	36°42	44°49	51°10	16°57	31°22	42°26	50°38	56°44
14°00	26°30	36°48	44°55	51°16	17°00	31°27	42°32	50°44	56°48

1 (17°00—23°00) 2 (31°27—40°20) 3 (42°32—51°51) 4 (50°44—59°30) 5 (56°48—64°46)

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17°00	31°27	42°32	50°44	56°48	20°00	36°03	47°31	55°31	61°13
17°03	31°31	42°37	50°49	56°53	20°03	36°08	47°36	55°35	61°17
17°06	31°36	42°42	50°54	56°58	20°06	36°12	47°40	55°40	61°21
17°09	31°41	42°48	50°59	57°03	20°09	36°16	47°45	55°44	61°24
17°12	31°46	42°53	51°04	57°08	20°12	36°21	47°49	55°48	61°28
17°15	31°50	42°58	51°10	57°13	20°15	36°25	47°54	55°53	61°32
17°18	31°55	43°03	51°15	57°18	20°18	36°30	47°59	55°57	61°36
17°21	32°00	43°09	51°20	57°23	20°21	36°34	48°03	56°01	61°40
17°24	32°05	43°14	51°25	57°27	20°24	36°39	48°08	56°05	61°44
17°27	32°09	43°19	51°30	57°32	20°27	36°43	48°12	56°10	61°48
17°30	32°14	43°24	51°35	57°37	20°30	36°47	48°17	56°14	61°51
17°33	32°19	43°30	51°40	57°41	20°33	36°52	48°21	56°18	61°55
17°36	32°24	43°35	51°46	57°46	20°36	36°56	48°26	56°22	61°59
17°39	32°28	43°40	51°51	57°51	20°39	37°00	48°30	56°26	62°03
17°42	32°33	43°45	51°56	57°56	20°42	37°05	48°35	56°31	62°07
17°45	32°38	43°50	52°01	58°00	20°45	37°09	48°40	56°35	62°10
17°48	32°42	43°56	52°06	58°05	20°48	37°13	48°44	56°39	62°14
17°51	32°47	44°01	52°11	58°09	20°51	37°18	48°48	56°43	62°18
17°54	32°52	44°06	52°16	58°14	20°54	37°22	48°53	56°47	62°21
17°57	32°56	44°11	52°21	58°19	20°57	37°27	48°57	56°51	62°25
18°00	33°01	44°16	52°25	58°23	21°00	37°31	49°02	56°55	62°29
18°03	33°06	44°21	52°30	58°28	21°03	37°35	49°06	57°00	62°32
18°06	33°10	44°26	52°35	58°32	21°06	37°40	49°11	57°04	62°36
18°09	33°15	44°31	52°40	58°37	21°09	37°44	49°15	57°08	62°40
18°12	33°20	44°36	52°45	58°41	21°12	37°48	49°19	57°12	62°43
18°15	33°24	44°41	52°50	58°46	21°15	37°52	49°24	57°16	62°47
18°18	33°29	44°46	52°55	58°50	21°18	37°57	49°28	57°20	62°51
18°21	33°34	44°51	53°00	58°55	21°21	38°01	49°33	57°24	62°54
18°24	33°38	44°57	53°04	58°59	21°24	38°05	49°37	57°28	62°58
18°27	33°43	45°02	53°09	59°04	21°27	38°10	49°41	57°32	63°01
18°30	33°47	45°07	53°14	59°08	21°30	38°14	49°46	57°36	63°05
18°33	33°52	45°11	53°19	59°12	21°33	38°18	49°50	57°40	63°08
18°36	33°57	45°16	53°24	59°17	21°36	38°22	49°54	57°44	63°12
18°39	34°01	45°21	53°28	59°21	21°39	38°27	49°59	57°48	63°15
18°42	34°06	45°26	53°33	59°25	21°42	38°31	50°03	57°52	63°19
18°45	34°10	45°31	53°38	59°30	21°45	38°35	50°07	57°56	63°22
18°48	34°15	45°36	53°42	59°34	21°48	38°39	50°12	58°00	63°26
18°51	34°20	45°41	53°47	59°38	21°51	38°44	50°16	58°03	63°29
18°54	34°24	45°46	53°52	59°43	21°54	38°48	50°20	58°07	63°33
18°57	34°29	45°51	53°56	59°47	21°57	38°52	50°24	58°11	63°36
19°00	34°33	45°56	54°01	59°51	22°00	38°56	50°29	58°15	63°40
19°03	34°38	46°01	54°06	59°55	20°03	39°01	50°33	58°19	63°43
19°06	34°42	46°05	54°10	60°00	22°06	39°05	50°37	58°23	63°47
19°09	34°47	46°10	54°15	60°04	22°09	39°09	50°41	58°27	63°50
19°12	34°51	46°15	54°20	60°08	22°12	39°13	50°45	58°30	63°53
19°15	34°56	46°20	54°24	60°12	22°15	39°17	50°50	58°34	63°57
19°18	35°00	46°25	54°29	60°16	22°18	39°22	50°54	58°38	64°00
19°21	35°05	46°30	54°33	60°20	22°21	39°26	50°58	58°42	64°04
19°24	35°09	46°34	54°38	60°24	22°24	39°30	51°02	58°46	64°07
19°27	35°14	46°39	54°42	60°28	22°27	39°34	51°06	58°49	64°10
19°30	35°18	46°44	54°47	60°33	22°30	39°38	51°10	58°53	64°14
19°33	35°23	46°49	54°51	60°37	22°33	39°43	51°15	58°57	64°17
19°36	35°27	46°53	54°56	60°41	22°36	39°47	51°19	59°01	64°20
19°39	35°32	46°58	55°00	60°45	22°39	39°51	51°23	59°04	64°24
19°42	35°36	47°03	55°05	60°49	22°42	39°55	51°27	59°08	64°27
19°45	35°41	47°08	55°09	60°53	22°45	39°59	51°31	59°12	64°30
19°48	35°45	47°12	55°13	60°57	22°48	40°03	51°35	59°16	64°33
19°51	35°50	47°17	55°18	61°01	22°51	40°07	51°39	59°19	64°37
19°54	35°54	47°22	55°22	61°05	22°54	40°12	51°43	59°23	64°40
19°57	35°59	47°26	55°27	61°09	22°57	40°16	51°47	59°26	64°43
20°00	36°03	47°31	55°31	61°13	23°00	40°20	51°51	59°30	64°46

1 (23°00—29°00) 2 (40°20—47°57) 3 (51°51—58°59) 4 (59°30—65°43) 5 (64°46—70°10)

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23°00	40°20	51°51	59°30	64°46	26°00	44°17	55°39	62°52	67°42
23°03	40°24	51°56	59°34	64°50	26°03	44°21	55°43	62°55	67°45
23°06	40°28	52°00	59°37	64°53	26°06	44°25	55°46	62°58	67°48
23°09	40°32	52°04	59°41	64°56	26°09	44°29	55°50	63°01	67°50
23°12	40°36	52°08	59°45	64°59	26°12	44°32	55°53	63°04	67°53
23°15	40°40	52°12	59°48	65°02	26°15	44°36	55°57	63°07	67°55
23°18	40°44	52°16	59°52	65°05	26°18	44°40	56°00	63°10	67°58
23°21	40°48	52°20	59°55	65°09	26°21	44°44	56°04	63°13	68°01
23°24	40°53	52°24	59°59	65°12	26°24	44°48	56°07	63°16	68°03
23°27	40°57	52°28	60°03	65°15	26°27	44°51	56°11	63°19	68°06
23°30	41°01	52°32	60°06	65°18	26°30	44°55	56°14	63°22	68°09
23°33	41°05	52°35	60°10	65°21	26°33	44°59	56°18	63°25	68°11
23°36	41°09	52°39	60°13	65°24	26°36	45°03	56°21	63°28	68°14
23°39	41°13	52°43	60°17	65°27	26°39	45°06	56°24	63°31	68°16
23°42	41°17	52°47	60°20	65°30	26°42	45°10	56°28	63°34	68°19
23°45	41°21	52°51	60°24	65°33	26°45	45°14	56°31	63°37	68°22
23°48	41°25	52°55	60°27	65°36	26°48	45°18	56°35	63°40	68°24
23°51	41°29	52°59	60°31	65°39	26°51	45°21	56°38	63°43	68°27
23°54	41°33	53°03	60°34	65°43	26°54	45°25	56°42	63°46	68°29
23°57	41°37	53°07	60°38	65°46	26°57	45°29	56°45	63°49	68°32
24°00	41°41	53°11	60°41	65°49	27°00	45°32	56°48	63°52	68°34
24°03	41°45	53°15	60°45	65°52	27°03	45°36	56°52	63°55	68°37
24°06	41°49	53°18	60°48	65°55	27°06	45°40	56°55	63°58	68°39
24°09	41°53	53°22	60°51	65°58	27°09	45°44	56°59	64°01	68°42
24°12	41°57	53°26	60°55	66°01	27°12	45°47	57°02	64°04	68°44
24°15	42°01	53°30	60°58	66°04	27°15	45°51	57°05	64°06	68°47
24°18	42°05	53°34	61°02	66°07	27°18	45°55	57°09	64°09	68°49
24°21	42°09	53°38	61°05	66°10	27°21	45°58	57°12	64°12	68°52
24°24	42°13	53°41	61°08	66°13	27°24	46°02	57°15	64°15	68°54
24°27	42°17	53°45	61°12	66°15	27°27	46°06	57°19	64°18	68°56
24°30	42°21	53°49	61°15	66°18	27°30	46°09	57°22	64°21	68°59
24°33	42°25	53°53	61°18	66°21	27°33	46°13	57°25	64°24	69°01
24°36	42°29	53°57	61°22	66°24	27°36	46°17	57°29	64°27	69°04
24°39	42°33	54°00	61°25	66°27	27°39	46°20	57°32	64°29	69°06
24°42	42°37	54°04	61°28	66°30	27°42	46°24	57°35	64°32	69°09
24°45	42°41	54°08	61°32	66°33	27°45	46°28	57°39	64°35	69°11
24°48	42°45	54°12	61°35	66°36	27°48	46°31	57°42	64°38	69°14
24°51	42°48	54°15	61°38	66°39	27°51	46°35	57°45	64°41	69°16
24°54	42°52	54°19	61°42	66°41	27°54	46°38	57°48	64°43	69°18
24°57	42°56	54°23	61°45	66°44	27°57	46°42	57°52	64°46	69°21
25°00	43°00	54°27	61°48	66°47	28°00	46°46	57°55	64°49	69°23
25°03	43°04	54°30	61°51	66°50	28°03	46°49	57°58	64°52	69°26
25°06	43°08	54°34	61°55	66°53	28°06	46°53	58°01	64°55	69°28
25°09	43°12	54°38	61°58	66°56	28°09	46°56	58°05	64°57	69°30
25°12	43°16	54°41	62°01	66°58	28°12	47°00	58°08	65°00	69°33
25°15	43°20	54°45	62°04	67°01	28°15	47°04	58°11	65°03	69°35
25°18	43°24	54°49	62°08	67°04	28°18	47°07	58°14	65°06	69°37
25°21	43°27	54°52	62°11	67°07	28°21	47°11	58°18	65°08	69°40
25°24	43°31	54°56	62°14	67°10	28°24	47°14	58°21	65°11	69°42
25°27	43°35	55°00	62°17	67°12	28°27	47°18	58°24	65°14	69°44
25°30	43°39	55°03	62°20	67°15	28°30	47°22	58°27	65°17	69°47
25°33	43°43	55°07	62°24	67°18	28°33	47°25	58°30	65°19	69°49
25°36	43°47	55°11	62°27	67°21	28°36	47°29	58°34	65°22	69°51
25°39	43°51	55°14	62°30	67°23	28°39	47°32	58°37	65°25	69°54
25°42	43°54	55°18	62°33	67°26	28°42	47°36	58°40	65°27	69°56
25°45	43°58	55°21	62°36	67°29	28°45	47°39	58°43	65°30	69°58
25°48	44°02	55°25	62°39	67°32	28°48	47°43	58°46	65°33	70°01
25°51	44°06	55°28	62°42	67°34	28°51	47°46	58°49	65°35	70°03
25°54	44°10	55°32	62°45	67°37	28°54	47°50	58°53	65°38	70°05
25°57	44°13	55°35	62°49	67°40	28°57	47°53	58°56	65°41	70°07
26°00	44°17	55°39	62°52	67°42	29°00	47°57	58°59	65°43	70°10

1 (29°00—35°00) 2 (47°57—54°28) 3 (58°59—64°33) 4 (65°43—70°21) 5 (70°10—74°04)

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29°00	47°57	58°59	65°43	70°10	32°00	51°20	61°55	68°12	72°15
29°03	48°00	59°02	65°46	70°12	32°03	51°23	61°58	68°14	72°17
29°06	48°04	59°05	65°49	70°14	32°06	51°27	62°01	68°16	72°19
29°09	48°07	59°08	65°51	70°16	32°09	51°30	62°04	68°19	72°21
29°12	48°11	59°11	65°54	70°19	32°12	51°33	62°06	68°21	72°23
29°15	48°14	59°14	65°57	70°21	32°15	51°36	62°09	68°23	72°25
29°18	48°18	59°17	65°59	70°23	32°18	51°39	62°12	68°25	72°27
29°21	48°21	59°20	66°02	70°25	32°21	51°43	62°15	68°28	72°29
29°24	48°25	59°24	66°04	70°27	32°24	51°46	62°17	68°30	72°31
29°27	48°28	59°27	66°07	70°30	32°27	51°49	62°20	68°32	72°32
29°30	48°32	59°30	66°10	70°32	32°30	51°52	62°23	68°34	72°34
29°33	48°35	59°33	66°12	70°34	32°33	51°56	62°26	68°37	72°36
29°36	48°39	59°36	66°15	70°36	32°36	51°59	62°28	68°39	72°38
29°39	48°42	59°39	66°17	70°38	32°39	52°02	62°31	68°41	72°40
29°42	48°46	59°42	66°20	70°41	32°42	52°05	62°34	68°43	72°42
29°45	48°49	59°45	66°23	70°43	32°45	52°08	62°36	68°46	72°44
29°48	48°53	59°48	66°25	70°45	32°48	52°12	62°39	68°48	72°46
29°51	48°56	59°51	66°28	70°47	32°51	52°15	62°42	68°50	72°47
29°54	49°00	59°54	66°30	70°49	32°54	52°18	62°44	68°52	72°49
29°57	49°03	59°57	66°33	70°51	32°57	52°21	62°47	68°54	72°51
30°00	49°06	60°00	66°35	70°54	33°00	52°24	62°50	68°57	72°53
30°03	49°10	60°03	66°38	70°56	33°03	52°28	62°52	68°59	72°55
30°06	49°13	60°06	66°40	70°58	33°06	52°31	62°55	69°01	72°57
30°09	49°17	60°09	66°43	71°00	33°09	52°34	62°58	69°03	72°58
30°12	49°20	60°12	66°45	71°02	33°12	52°37	63°00	69°05	73°00
30°15	49°23	60°15	66°48	71°04	33°15	52°40	63°03	69°07	73°02
30°18	49°27	60°18	66°50	71°06	33°18	52°43	63°06	69°10	73°04
30°21	49°30	60°21	66°53	71°08	33°21	52°46	63°08	69°12	73°06
30°24	49°34	60°24	66°55	71°11	33°24	52°50	63°11	69°14	73°08
30°27	49°37	60°27	66°58	71°13	33°27	52°53	63°14	69°16	73°09
30°30	49°40	60°30	67°00	71°15	33°30	52°56	63°16	69°18	73°11
30°33	49°44	60°33	67°03	71°17	33°33	52°59	63°19	69°21	73°13
30°36	49°47	60°36	67°05	71°19	33°36	53°02	63°21	69°23	73°15
30°39	49°51	60°39	67°08	71°21	33°39	53°05	63°24	69°25	73°17
30°42	49°54	60°41	67°10	71°23	33°42	53°08	63°27	69°27	73°18
30°45	49°57	60°44	67°12	71°25	33°45	53°12	63°29	69°29	73°20
30°48	50°01	60°47	67°15	71°27	33°48	53°15	63°32	69°31	73°22
30°51	50°04	60°50	67°17	71°29	33°51	53°18	63°34	69°33	73°24
30°54	50°07	60°53	67°20	71°31	33°54	53°21	63°37	69°36	73°26
30°57	50°11	60°56	67°22	71°33	33°57	53°24	63°40	69°38	73°27
31°00	50°14	60°59	67°24	71°35	34°00	53°27	63°42	69°40	73°29
31°03	50°17	61°02	67°27	71°37	34°03	53°30	63°45	69°42	73°31
31°06	50°21	61°05	67°29	71°39	34°06	53°33	63°47	69°44	73°33
31°09	50°24	61°07	67°32	71°41	34°09	53°36	63°50	69°46	73°34
31°12	50°27	61°10	67°34	71°43	34°12	53°39	63°52	69°48	73°36
31°15	50°31	61°13	67°37	71°45	34°15	53°42	63°55	69°50	73°38
31°18	50°34	61°16	67°39	71°47	34°18	53°46	63°58	69°52	73°40
31°21	50°37	61°19	67°41	71°49	34°21	53°49	64°00	69°54	73°41
31°24	50°41	61°22	67°43	71°51	34°24	53°52	64°03	69°57	73°43
31°27	50°44	61°25	67°46	71°53	34°27	53°55	64°05	69°59	73°45
31°30	50°47	61°27	67°48	71°55	34°30	53°58	64°08	70°01	73°46
31°33	50°51	61°30	67°51	71°57	34°33	54°01	64°10	70°03	73°48
31°36	50°54	61°33	67°53	71°59	34°36	54°04	64°13	70°05	73°50
31°39	50°57	61°36	67°55	72°01	34°39	54°07	64°15	70°07	73°52
31°42	51°00	61°39	67°58	72°03	34°42	54°10	64°18	70°09	73°53
31°45	51°04	61°41	68°00	72°05	34°45	54°13	64°20	70°11	73°55
31°48	51°07	61°44	68°02	72°07	34°48	54°16	64°23	70°13	73°57
31°51	51°10	61°47	68°05	72°09	34°51	54°19	64°25	70°15	73°58
31°54	51°14	61°50	68°07	72°11	34°54	54°22	64°28	70°17	74°00
31°57	51°17	61°53	68°09	72°13	34°57	54°25	64°30	70°19	74°02
32°00	51°20	61°55	68°12	72°15	35°00	54°28	64°33	70°21	74°04

1 (35°00—41°00) 2 (54°28—60°06) 3 (64°33—69°01) 4 (70°21—73°57) 5 (74°04—77°03)

1	2	3	4	5	1	2	3	4	5
35°00	54°28	64°33	70°21	74°04	38°00	57°23	66°54	72°15	75°38
35°03	54°31	64°35	70°23	74°05	38°03	57°26	66°56	72°17	75°40
35°06	54°34	64°38	70°25	74°07	38°06	57°29	66°58	72°19	75°42
35°09	54°37	64°40	70°27	74°09	38°09	57°31	67°00	72°21	75°43
35°12	54°40	64°43	70°29	74°10	38°12	57°34	67°03	72°23	75°44
35°15	54°43	64°45	70°31	74°12	38°15	57°37	67°05	72°24	75°46
35°18	54°46	64°48	70°33	74°14	38°18	57°40	67°07	72°26	75°47
35°21	54°49	64°50	70°35	74°15	38°21	57°43	67°09	72°28	75°49
35°24	54°52	64°53	70°37	74°17	38°24	57°45	67°11	72°30	75°50
35°27	54°55	64°55	70°39	74°19	38°27	57°48	67°14	72°31	75°52
35°30	54°58	64°57	70°41	74°20	38°30	57°51	67°16	72°33	75°53
35°33	55°01	65°00	70°43	74°22	38°33	57°54	67°18	72°35	75°55
35°36	55°04	65°02	70°45	74°24	38°36	57°56	67°20	72°37	75°56
35°39	55°07	65°04	70°47	74°25	38°39	57°59	67°22	72°38	75°58
35°42	55°10	65°07	70°49	74°27	38°42	58°02	67°25	72°40	75°59
35°45	55°13	65°09	70°51	74°28	38°45	58°05	67°27	72°42	76°00
35°48	55°16	65°11	70°53	74°30	38°48	58°07	67°29	72°44	76°02
35°51	55°19	65°14	70°55	74°32	38°51	58°10	67°31	72°45	76°03
35°54	55°22	65°17	70°57	74°33	38°54	58°13	67°33	72°47	76°05
35°57	55°25	65°19	70°59	74°35	38°57	58°16	67°35	72°49	76°06
36°00	55°28	65°21	71°01	74°37	39°00	58°18	67°38	72°51	76°08
36°03	55°31	65°24	71°03	74°38	39°03	58°21	67°40	72°52	76°09
36°06	55°34	65°26	71°05	74°40	39°06	58°24	67°42	72°54	76°10
36°09	55°37	65°28	71°07	74°41	39°09	58°27	67°44	72°56	76°12
36°12	55°40	65°31	71°08	74°43	39°12	58°29	67°46	72°57	76°13
36°15	55°43	65°33	71°10	74°45	39°15	58°32	67°48	72°59	76°15
36°18	55°45	65°36	71°12	74°46	39°18	58°35	67°50	73°01	76°16
36°21	55°48	65°38	71°14	74°48	39°21	58°38	67°53	73°03	76°18
36°24	55°51	65°40	71°16	74°49	39°24	58°40	67°55	73°04	76°19
36°27	55°54	65°43	71°18	74°51	39°27	58°43	67°57	73°06	76°20
36°30	55°57	65°45	71°20	74°53	39°30	58°46	67°59	73°08	76°22
36°33	56°00	65°47	71°22	74°54	39°33	58°48	68°01	73°09	76°23
36°36	56°03	65°50	71°24	74°56	39°36	58°51	68°03	73°11	76°25
36°39	56°06	65°52	71°26	74°57	39°39	58°54	68°05	73°13	76°26
36°42	56°09	65°54	71°28	74°59	39°42	58°56	68°07	73°14	76°27
36°45	56°12	65°57	71°29	75°00	39°45	58°59	68°10	73°16	76°29
36°48	56°15	65°59	71°31	75°02	39°48	59°02	68°12	73°18	76°30
36°51	56°17	66°01	71°33	75°04	39°51	59°05	68°14	73°19	76°31
36°54	56°20	66°04	71°35	75°05	39°54	59°07	68°16	73°21	76°33
36°57	56°23	66°06	71°37	75°07	39°57	59°10	68°18	73°23	76°34
37°00	56°26	66°08	71°39	75°08	40°00	59°13	68°20	73°25	76°36
37°03	56°29	66°11	71°41	75°10	40°03	59°15	68°22	73°26	76°37
37°06	56°32	66°13	71°42	75°11	40°06	59°18	68°24	73°28	76°38
37°09	56°35	66°15	71°44	75°13	40°09	59°21	68°26	73°30	76°40
37°12	56°38	66°17	71°46	75°14	40°12	59°23	68°28	73°31	76°41
37°15	56°40	66°20	71°48	75°16	40°15	59°26	68°30	73°33	76°42
37°18	56°43	66°22	71°50	75°17	40°18	59°29	68°33	73°35	76°44
37°21	56°46	66°24	71°52	75°19	40°21	59°31	68°35	73°36	76°45
37°24	56°49	66°27	71°54	75°20	40°24	59°34	68°37	73°38	76°47
37°27	56°52	66°29	71°55	75°22	40°27	59°37	68°39	73°39	76°48
37°30	56°55	66°31	71°57	75°23	40°30	59°39	68°41	73°41	76°49
37°33	56°58	66°33	71°59	75°25	40°33	59°42	68°43	73°43	76°51
37°36	57°00	66°36	72°01	75°27	40°36	59°45	68°45	73°44	76°52
37°39	57°03	66°38	72°03	75°28	40°39	59°47	68°47	73°46	76°53
37°42	57°06	66°40	72°05	75°30	40°42	59°50	68°49	73°48	76°55
37°45	57°09	66°42	72°06	75°31	40°45	59°52	68°51	73°49	76°56
37°48	57°12	66°45	72°08	75°33	40°48	59°55	68°53	73°51	76°57
37°51	57°14	66°47	72°10	75°34	40°51	59°58	68°55	73°52	76°59
37°54	57°17	66°49	72°12	75°36	40°54	60°01	68°57	73°54	77°00
37°57	57°20	66°51	72°14	75°37	40°57	60°03	68°59	73°56	77°01
38°00	57°23	66°54	72°15	75°38	41°00	60°06	69°01	73°57	77°03

1 (41°00—47°00) 2 (60°06—65°00) 3 (69°01—72°44) 4 (73°57—76°53) 5 (77°03—79°26)

1	2	3	4	5	1	2	3	4	5
41°00	60°06	69°01	73°57	77°03	44°00	62°38	70°57	75°29	78°18
41°03	60°08	69°03	73°59	77°04	44°03	62°40	70°59	75°31	78°19
41°06	60°11	69°05	74°01	77°05	44°06	62°42	71°01	75°32	78°20
41°09	60°14	69°07	74°02	77°07	44°09	62°45	71°03	75°34	78°21
41°12	60°16	69°09	74°04	77°08	44°12	62°47	71°05	75°35	78°23
41°15	60°19	69°11	74°05	77°09	44°15	62°50	71°07	75°36	78°24
41°18	60°21	69°13	74°07	77°10	44°18	62°52	71°08	75°38	78°25
41°21	60°24	69°15	74°09	77°12	44°21	62°55	71°10	75°39	78°26
41°24	60°26	69°17	74°10	77°13	44°24	62°57	71°12	75°41	78°27
41°27	60°29	69°19	74°12	77°14	44°27	63°00	71°14	75°42	78°29
41°30	60°31	69°21	74°13	77°16	44°30	63°02	71°16	75°44	78°30
41°33	60°34	69°23	74°15	77°17	44°33	63°04	71°18	75°45	78°31
41°36	60°37	69°25	74°16	77°18	44°36	63°07	71°19	75°46	78°32
41°39	60°39	69°27	74°18	77°20	44°39	63°09	71°21	75°48	78°33
41°42	60°42	69°29	74°20	77°21	44°42	63°12	71°23	75°49	78°34
41°45	60°44	69°31	74°21	77°22	44°45	63°14	71°25	75°51	78°35
41°48	60°47	69°33	74°23	77°23	44°48	63°16	71°27	75°52	78°37
41°51	60°50	69°35	74°24	77°25	44°51	63°19	71°28	75°54	78°38
41°54	60°52	69°37	74°26	77°26	44°54	63°21	71°30	75°55	78°39
41°57	60°55	69°39	74°27	77°27	44°57	63°24	71°32	75°56	78°40
42°00	60°57	69°41	74°29	77°29	45°00	63°26	71°34	75°58	78°41
42°03	61°00	69°43	74°31	77°30	45°03	63°29	71°36	75°59	78°43
42°06	61°02	69°45	74°32	77°31	45°06	63°31	71°38	76°01	78°44
42°09	61°05	69°47	74°34	77°32	45°09	63°33	71°39	76°02	78°45
42°12	61°08	69°49	74°35	77°34	45°12	63°36	71°41	76°03	78°46
42°15	61°10	69°51	74°37	77°35	45°15	63°38	71°43	76°05	78°47
42°18	61°13	69°53	74°38	77°36	45°18	63°40	71°45	76°06	78°48
42°21	61°15	69°55	74°40	77°37	45°21	63°43	71°46	76°08	78°49
42°24	61°18	69°57	74°41	77°39	45°24	63°45	71°48	76°09	78°51
42°27	61°20	69°59	74°43	77°40	45°27	63°48	71°50	76°10	78°52
42°30	61°23	70°01	74°44	77°41	45°30	63°50	71°52	76°12	78°53
42°33	61°25	70°03	74°46	77°43	45°33	63°52	71°54	76°13	78°54
42°36	61°28	70°04	74°47	77°44	45°36	63°55	71°55	76°15	78°55
42°39	61°30	70°06	74°49	77°45	45°39	63°57	71°57	76°16	78°56
42°42	61°33	70°08	74°50	77°46	45°42	63°59	71°59	76°17	78°57
42°45	61°35	70°10	74°52	77°48	45°45	64°02	72°01	76°19	78°59
42°48	61°38	70°12	74°54	77°49	45°48	64°04	72°02	76°20	79°00
42°51	61°40	70°14	74°55	77°50	45°51	64°07	72°04	76°22	79°01
42°54	61°43	70°16	74°57	77°51	45°54	64°09	72°06	76°23	79°02
42°57	61°46	70°18	74°58	77°52	45°57	64°11	72°08	76°24	79°03
43°00	61°48	70°20	75°00	77°54	46°00	64°14	72°09	76°26	79°04
43°03	61°50	70°22	75°01	77°55	46°03	64°16	72°11	76°27	79°05
43°06	61°53	70°24	75°03	77°56	46°06	64°18	72°13	76°28	79°06
43°09	61°56	70°26	75°04	77°57	46°09	64°21	72°15	76°30	79°07
43°12	61°58	70°27	75°06	77°59	46°12	64°23	72°16	76°31	79°09
43°15	62°01	70°29	75°07	78°00	46°15	64°25	72°18	76°32	79°10
43°18	62°03	70°31	75°09	78°01	46°18	64°28	72°20	76°34	79°11
43°21	62°06	70°33	75°10	78°02	46°21	64°30	72°22	76°35	79°12
43°24	62°08	70°35	75°11	78°03	46°24	64°32	72°23	76°36	79°13
43°27	62°10	70°37	75°13	78°05	46°27	64°35	72°25	76°38	79°14
43°30	62°13	70°39	75°14	78°06	46°30	64°37	72°27	76°39	79°15
43°33	62°15	70°41	75°16	78°07	46°33	64°39	72°29	76°41	79°16
43°36	62°18	70°43	75°17	78°08	46°36	64°42	72°30	76°42	79°17
43°39	62°20	70°44	75°19	78°10	46°39	64°44	72°32	76°43	79°19
43°42	62°23	70°46	75°20	78°11	46°42	64°46	72°34	76°45	79°20
43°45	62°25	70°48	75°22	78°12	46°45	64°49	72°35	76°46	79°21
43°48	62°28	70°50	75°23	78°13	46°48	64°51	72°37	76°47	79°22
43°51	62°30	70°52	75°25	78°14	46°51	64°53	72°39	76°49	79°23
43°54	62°33	70°54	75°26	78°16	46°54	64°55	72°41	76°50	79°24
43°57	62°35	70°56	75°28	78°17	46°57	64°58	72°42	76°51	79°25
44°00	62°38	70°57	75°29	78°18	47°00	65°00	72°44	76°53	79°26

1 (47°00—53°00) 2 (65°00—69°21) 3 (72°44—75°54) 4 (76°53—79°20) 5 (79°26—81°26)

1	2	3	4	5	1	2	3	4	5
47°00	65°00	72°44	76°53	79°26	50°00	67°14	74°22	78°09	80°28
47°03	65°02	72°46	76°54	79°27	50°03	67°17	74°24	78°10	80°29
47°06	65°05	72°47	76°55	79°28	50°06	67°19	74°26	78°12	80°30
47°09	65°07	72°49	76°57	79°29	50°09	67°21	74°27	78°13	80°31
47°12	65°09	72°51	76°58	79°30	50°12	67°23	74°29	78°14	80°32
47°15	65°12	72°52	76°59	79°32	50°15	67°25	74°30	78°15	80°33
47°18	65°14	72°54	77°01	79°33	50°18	67°27	74°32	78°16	80°34
47°21	65°16	72°56	77°02	79°34	50°21	67°30	74°33	78°18	80°35
47°24	65°18	72°58	77°03	79°35	50°24	67°32	74°35	78°19	80°36
47°27	65°21	72°59	77°05	79°36	50°27	67°34	74°36	78°20	80°37
47°30	65°23	73°01	77°06	79°37	50°30	67°36	74°38	78°21	80°38
47°33	65°25	73°03	77°07	79°38	50°33	67°38	74°40	78°22	80°39
47°36	65°28	73°04	77°09	79°39	50°36	67°40	74°41	78°24	80°40
47°39	65°30	73°06	77°10	79°40	50°39	67°43	74°43	78°25	80°41
47°42	65°32	73°08	77°11	79°41	50°42	67°45	74°44	78°26	80°42
47°45	65°34	73°09	77°12	79°42	50°45	67°47	74°46	78°27	80°43
47°48	65°36	73°11	77°14	79°43	50°48	67°49	74°47	78°29	80°44
47°51	65°39	73°13	77°15	79°44	50°51	67°51	74°49	78°30	80°45
47°54	65°41	73°14	77°16	79°45	50°54	67°53	74°50	78°31	80°46
47°57	65°44	73°16	77°18	79°46	50°57	67°55	74°52	78°32	80°47
48°00	65°46	73°18	77°19	79°48	51°00	67°57	74°54	78°33	80°48
48°03	65°48	73°19	77°20	79°49	51°03	67°59	74°55	78°35	80°49
48°06	65°50	73°21	77°21	79°50	51°06	68°02	74°57	78°36	80°50
48°09	65°53	73°23	77°23	79°51	51°09	68°04	74°58	78°37	80°51
48°12	65°55	73°24	77°24	79°52	51°12	68°06	75°00	78°38	80°52
48°15	65°57	73°26	77°25	79°53	51°15	68°08	75°01	78°39	80°53
48°18	65°59	73°28	77°27	79°54	51°18	68°10	75°03	78°40	80°54
48°21	66°02	73°29	77°28	79°55	51°21	68°12	75°04	78°42	80°55
48°24	66°04	73°31	77°29	79°56	51°24	68°14	75°06	78°43	80°56
48°27	66°06	73°32	77°30	79°57	51°27	67°17	75°07	78°44	80°57
48°30	66°08	73°34	77°32	79°58	51°30	68°19	75°09	78°45	80°58
48°33	66°10	73°36	77°33	79°59	51°33	68°21	75°11	78°46	80°59
48°36	66°13	73°37	77°34	80°00	51°36	68°23	75°12	78°48	81°00
48°39	66°15	73°39	77°35	80°01	51°39	68°25	75°14	78°49	81°01
48°42	66°17	73°41	77°37	80°02	51°42	68°27	75°15	78°50	81°01
48°45	66°19	73°42	77°38	80°03	51°45	68°29	75°17	78°51	81°02
48°48	66°22	73°44	77°39	80°04	51°48	68°31	75°18	78°52	81°03
48°51	66°24	73°46	77°41	80°05	51°51	68°33	75°20	78°53	81°04
48°54	66°26	73°47	77°42	80°06	51°54	68°36	75°21	78°54	81°05
48°57	66°28	73°49	77°43	80°07	51°57	68°38	75°23	78°56	81°06
49°00	66°30	73°50	77°44	80°08	52°00	68°40	75°24	78°57	81°07
49°03	66°33	73°52	77°46	80°09	52°03	68°42	75°26	78°58	81°08
49°06	66°35	73°54	77°47	80°10	52°06	68°44	75°27	78°59	81°09
49°09	66°37	73°55	77°48	80°11	52°09	68°46	75°29	79°00	81°10
49°12	66°39	73°57	77°49	80°12	52°12	68°48	75°31	79°02	81°11
49°15	66°42	73°58	77°51	80°13	52°15	68°50	75°32	79°03	81°12
49°18	66°44	74°00	77°52	80°14	52°18	68°52	75°33	79°04	81°13
49°21	66°46	74°02	77°53	80°15	52°21	68°54	75°35	79°05	81°14
49°24	66°48	74°03	77°54	80°16	52°24	68°56	75°36	79°06	81°15
49°27	66°50	74°05	77°56	80°17	52°27	68°59	75°38	79°07	81°16
49°30	66°53	74°07	77°57	80°18	52°30	69°01	75°39	79°08	81°17
49°33	66°55	74°08	77°58	80°19	52°33	69°03	75°41	79°10	81°18
49°36	66°57	74°10	78°00	80°20	52°36	69°05	75°42	79°11	81°18
49°39	66°59	74°11	78°01	80°21	52°39	69°07	75°44	79°12	81°19
49°42	67°01	74°13	78°02	80°22	52°42	69°09	75°45	79°13	81°20
49°45	67°03	74°14	78°03	80°23	52°45	69°11	75°46	79°14	81°21
49°48	67°06	74°16	78°04	80°24	52°48	69°13	75°48	79°15	81°22
49°51	67°08	74°18	78°05	80°25	52°51	69°15	75°50	79°16	81°23
49°54	67°10	74°19	78°07	80°26	52°54	69°17	75°51	79°18	81°24
49°57	67°12	74°21	78°08	80°27	52°57	69°19	75°52	79°19	81°25
50°00	67°14	74°22	78°09	80°28	53°00	69°21	75°54	79°20	81°26

1 (53°00—59°00) 2 (69°21—73°17) 3 (75°54—78°40) 4 (79°20—81°27) 5 (81°26—83°09)

1	2	3	4	5	1	2	3	4	5
53°00	69°21	75°54	79°20	81°26	56°00	71°22	77°20	80°26	82°19
53°03	69°23	75°55	79°21	81°27	56°03	71°24	77°21	80°27	82°20
53°06	69°25	75°57	79°22	81°28	56°06	71°26	77°22	80°28	82°21
53°09	69°27	75°58	79°23	81°29	56°09	71°28	77°24	80°29	82°22
53°12	69°29	76°00	79°24	81°29	56°12	71°30	77°25	80°30	82°22
53°15	69°32	76°01	79°25	81°30	56°15	71°32	77°27	80°31	82°23
53°18	69°34	76°03	79°27	81°31	56°18	71°34	77°28	80°32	82°24
53°21	69°36	76°04	79°28	81°32	56°21	71°35	77°29	80°33	82°25
53°24	69°38	76°06	79°29	81°33	56°24	71°37	77°31	80°34	82°26
53°27	69°40	76°07	79°30	81°34	56°27	71°39	77°32	80°35	82°27
53°30	69°42	76°09	79°31	81°35	56°30	71°41	77°34	80°36	82°28
53°33	69°44	76°10	79°32	81°36	56°33	71°43	77°35	80°37	82°28
53°36	69°46	76°12	79°33	81°37	56°36	71°45	77°36	80°38	82°29
53°39	69°48	76°13	79°35	81°38	56°39	71°47	77°38	80°39	82°30
53°42	69°50	76°14	79°36	81°39	56°42	71°49	77°39	80°40	82°31
53°45	69°52	76°16	79°37	81°39	56°45	71°51	77°40	80°41	82°32
53°48	69°54	76°17	79°38	81°40	56°48	71°53	77°42	80°43	82°33
53°51	69°56	76°19	79°39	81°41	56°51	71°55	77°43	80°44	82°33
53°54	69°58	76°20	79°40	81°42	56°54	71°57	77°44	80°45	82°34
53°57	70°00	76°22	79°41	81°43	56°57	71°59	77°46	80°46	82°35
54°00	70°02	76°23	79°42	81°44	57°00	72°01	77°47	80°47	82°36
54°03	70°04	76°25	79°43	81°45	57°03	72°03	77°48	80°48	82°37
54°06	70°06	76°26	79°45	81°46	57°06	72°05	77°50	80°49	82°38
54°09	70°08	76°27	79°46	81°47	57°09	72°06	77°51	80°50	82°38
54°12	70°10	76°29	79°47	81°48	57°12	72°08	77°53	80°51	82°39
54°15	70°12	76°30	79°48	81°48	57°15	72°10	77°54	80°52	82°40
54°18	70°14	76°32	79°49	81°49	57°18	72°12	77°55	80°53	82°41
54°21	70°16	76°33	79°50	81°50	57°21	72°14	77°57	80°54	82°42
54°24	70°18	76°35	79°51	81°51	57°24	72°16	77°58	80°55	82°43
54°27	70°20	76°36	79°52	81°52	57°27	72°18	77°59	80°56	82°43
54°30	70°22	76°38	79°53	81°53	57°30	72°20	78°01	80°57	82°44
54°33	70°24	76°39	79°54	81°54	57°33	72°22	78°02	80°58	82°45
54°36	70°26	76°40	79°56	81°55	57°36	72°24	78°03	80°59	82°46
54°39	70°28	76°42	79°57	81°56	57°39	72°26	78°05	81°00	82°47
54°42	70°30	76°43	79°58	81°56	57°42	72°28	78°06	81°01	82°48
54°45	70°32	76°45	79°59	81°57	57°45	72°29	78°07	81°02	82°48
54°48	70°34	76°46	80°00	81°58	57°48	72°31	78°09	81°03	82°49
54°51	70°36	76°47	80°01	81°59	57°51	72°33	78°10	81°04	82°50
54°54	70°38	76°49	80°02	82°00	57°54	72°35	78°11	81°05	82°51
54°57	70°40	76°50	80°03	82°01	57°57	72°37	78°13	81°06	82°52
55°00	70°42	76°52	80°04	82°02	58°00	72°39	78°14	81°07	82°53
55°03	70°44	76°53	80°05	82°03	58°03	72°41	78°15	81°08	82°53
55°06	70°46	76°54	80°06	82°03	58°06	72°43	78°17	81°09	82°54
55°09	70°48	76°56	80°08	82°04	58°09	72°45	78°18	81°10	82°55
55°12	70°50	76°57	80°09	82°05	58°12	72°47	78°19	81°11	82°56
55°15	70°52	76°59	80°10	82°06	58°15	72°48	78°21	81°12	82°57
55°18	70°54	77°00	80°11	82°07	58°18	72°50	78°22	81°13	82°58
55°21	70°56	77°02	80°12	82°08	58°21	72°52	78°23	81°14	82°58
55°24	70°58	77°03	80°13	82°09	58°24	72°54	78°25	81°15	82°59
55°27	71°00	77°04	80°14	82°10	58°27	72°56	78°26	81°16	83°00
55°30	71°02	77°06	80°15	82°10	58°30	72°58	78°27	81°17	83°01
55°33	71°04	77°07	80°16	82°11	58°33	73°00	78°29	81°18	83°02
55°36	71°06	77°09	80°17	82°12	58°36	73°02	78°30	81°19	83°02
55°39	71°08	77°10	80°18	82°13	58°39	73°04	78°31	81°20	83°03
55°42	71°10	77°11	80°19	82°14	58°42	73°05	78°33	81°21	83°04
55°45	71°12	77°13	80°20	82°15	58°45	73°07	78°34	81°22	83°05
55°48	71°14	77°14	80°21	82°16	58°48	73°09	78°35	81°23	83°06
55°51	71°16	77°16	80°23	82°16	58°51	73°11	78°37	81°24	83°06
55°54	71°18	77°17	80°24	82°17	58°54	73°13	78°38	81°25	83°07
55°57	71°20	77°18	80°25	82°18	58°57	73°15	78°39	81°26	83°08
56°00	71°22	77°20	80°26	82°19	59°00	73°17	78°40	81°27	83°09

1 (59°00—65°00) 2 (73°17—76°52) 3 (78°40—81°10) 4 (81°27—83°21) 5 (83°09—84°40)

1	2	3	4	5	1	2	3	4	5
59°00	73°17	78°40	81°27	83°09	62°00	75°07	79°57	82°26	83°56
59°03	73°19	78°42	81°28	83°10	62°03	75°09	79°58	82°27	83°57
59°06	73°20	78°43	81°29	83°10	62°06	75°10	79°59	82°28	83°57
59°09	73°22	78°44	81°30	83°11	62°09	75°12	80°01	82°29	83°58
59°12	73°24	78°46	81°31	83°12	62°12	75°14	80°02	82°29	83°59
59°15	73°26	78°47	81°32	83°13	62°15	75°16	80°03	82°30	84°00
59°18	73°28	78°48	81°33	83°14	62°18	75°17	80°04	82°31	84°00
59°21	73°30	78°50	81°34	83°14	62°21	75°19	80°06	82°32	84°01
59°24	73°32	78°51	81°35	83°15	62°24	75°21	80°07	82°33	84°02
59°27	73°33	78°52	81°36	83°16	62°27	75°23	80°08	82°34	84°03
59°30	73°35	78°53	81°37	83°17	62°30	75°25	80°09	82°35	84°03
59°33	73°37	78°55	81°38	83°18	62°33	75°26	80°11	82°36	84°04
59°36	73°39	78°56	81°39	83°18	62°36	75°28	80°12	82°37	84°05
59°39	73°41	78°57	81°40	83°19	62°39	75°30	80°13	82°38	84°06
59°42	73°43	78°59	81°41	83°20	62°42	75°32	80°14	82°39	84°06
59°45	73°45	79°00	81°42	83°21	62°45	75°34	80°15	82°40	84°07
59°48	73°46	79°01	81°43	83°22	62°48	75°35	80°17	82°41	84°08
59°51	73°48	79°03	81°44	83°22	62°51	75°37	80°18	82°42	84°09
59°54	73°50	79°04	81°45	83°23	62°54	75°39	80°19	82°43	84°09
59°57	73°52	79°05	81°46	83°24	62°57	75°41	80°20	82°44	84°10
60°00	73°54	79°06	81°47	83°25	63°00	75°42	80°22	82°44	84°11
60°03	73°56	79°08	81°48	83°26	63°03	75°44	80°23	82°45	84°12
60°06	73°58	79°09	81°49	83°27	63°06	75°46	80°24	82°46	84°12
60°09	73°59	79°10	81°50	83°27	63°09	75°48	80°25	82°47	84°13
60°12	74°01	79°12	81°51	83°28	63°12	75°50	80°27	82°48	84°14
60°15	74°03	79°13	81°52	83°29	63°15	75°51	80°28	82°49	84°15
60°18	74°05	79°14	81°53	83°29	63°18	75°53	80°29	82°50	84°15
60°21	74°07	79°15	81°54	83°30	63°21	75°55	80°30	82°51	84°16
60°24	74°09	79°17	81°55	83°31	63°24	75°57	80°31	82°52	84°17
60°27	74°10	79°18	81°56	83°32	63°27	75°58	80°33	82°53	84°18
60°30	74°12	79°19	81°57	83°33	63°30	76°00	80°34	82°54	84°18
60°33	74°14	79°20	81°58	83°33	63°33	76°02	80°35	82°55	84°19
60°36	74°16	79°22	81°59	83°34	63°36	76°04	80°36	82°56	84°20
60°39	74°18	79°23	82°00	83°35	63°39	76°05	80°37	82°56	84°21
60°42	74°20	79°24	82°01	83°36	63°42	76°07	80°39	82°57	84°21
60°45	74°21	79°26	82°02	83°37	63°45	76°09	80°40	82°58	84°22
60°48	74°23	79°27	82°03	83°37	63°48	76°11	80°41	82°59	84°23
60°51	74°25	79°28	82°04	83°38	63°51	76°12	80°42	83°00	84°24
60°54	74°27	79°29	82°05	83°39	63°54	76°14	80°44	83°01	84°24
60°57	74°29	79°31	82°06	83°40	63°57	76°16	80°45	83°02	84°25
61°00	74°31	79°32	82°07	83°40	64°00	76°18	80°46	83°03	84°26
61°03	74°32	79°33	82°08	83°41	64°03	76°19	80°47	83°04	84°26
61°06	74°34	79°34	82°09	83°42	64°06	76°21	80°48	83°05	84°27
61°09	74°36	79°36	82°10	83°43	64°09	76°23	80°50	83°06	84°28
61°12	74°38	79°37	82°10	83°44	64°12	76°25	80°51	83°07	84°29
61°15	74°40	79°38	82°11	83°44	64°15	76°26	80°52	83°07	84°29
61°18	74°41	79°39	82°12	83°45	64°18	76°28	80°53	83°08	84°30
61°21	74°43	79°41	82°13	83°46	64°21	76°30	80°54	83°09	84°31
61°24	74°45	79°42	82°14	83°47	64°24	76°32	80°56	83°10	84°32
61°27	74°47	79°43	82°15	83°47	64°27	76°33	80°57	83°11	84°32
61°30	74°49	79°44	82°16	83°48	64°30	76°35	80°58	83°12	84°33
61°33	74°51	79°46	82°17	83°49	64°33	76°37	80°59	83°13	84°34
61°36	74°52	79°47	82°18	83°50	64°36	76°39	81°00	83°14	84°35
61°39	74°54	79°48	82°19	83°50	64°39	76°40	81°02	83°15	84°35
61°42	74°56	79°49	82°20	83°51	64°42	76°42	81°03	83°16	84°36
61°45	74°58	79°51	82°21	83°52	64°45	76°44	81°04	83°17	84°37
61°48	75°00	79°52	82°22	83°53	64°48	76°46	81°05	83°17	84°37
61°51	75°01	79°53	82°23	83°54	64°51	76°47	81°06	83°18	84°38
61°54	75°03	79°54	82°24	83°54	64°54	76°49	81°07	83°19	84°39
61°57	75°05	79°56	82°25	83°55	64°57	76°51	81°09	83°20	84°40
62°00	75°07	79°57	82°26	83°56	65°00	76°52	81°10	83°21	84°40

1 (65°00—74°00). 2 (76°52—81°50). 3 (81°10—84°32).

1	2	3	1	2	3	1	2	3
65°00	76°52	81°10	68°00	78°35	82°20	71°00	80°14	83°27
65°03	76°54	81°11	68°03	78°36	82°21	71°03	80°16	83°28
65°06	76°56	81°12	68°06	78°38	82°22	71°06	80°17	83°29
65°09	76°58	81°13	68°09	78°40	82°23	71°09	80°19	83°30
65°12	76°59	81°15	68°12	78°41	82°24	71°12	80°20	83°32
65°15	77°01	81°16	68°15	78°43	82°25	71°15	80°22	83°33
65°18	77°03	81°17	68°18	78°45	82°27	71°18	80°24	83°34
65°21	77°05	81°18	68°21	78°46	82°28	71°21	80°25	83°35
65°24	77°06	81°19	68°24	78°48	82°29	71°24	80°27	83°36
65°27	77°08	81°21	68°27	78°50	82°30	71°27	80°29	83°37
65°30	77°10	81°22	68°30	78°51	82°31	71°30	80°30	83°38
65°33	77°12	81°23	68°33	78°53	82°32	71°33	80°32	83°39
65°36	77°13	81°24	68°36	78°55	82°33	71°36	80°33	83°40
65°39	77°15	81°25	68°39	78°56	82°35	71°39	80°35	83°41
65°42	77°17	81°26	68°42	78°58	82°36	71°42	80°37	83°43
65°45	77°18	81°28	68°45	79°00	82°37	71°45	80°38	83°44
65°48	77°20	81°29	68°48	79°01	82°38	71°48	80°40	83°45
65°51	77°22	81°30	68°51	79°03	82°39	71°51	80°41	83°46
65°54	77°24	81°31	68°54	79°05	82°40	71°54	80°43	83°47
65°57	77°25	81°32	68°57	79°06	82°41	71°57	80°45	83°48
66°00	77°27	81°34	69°00	79°08	82°43	72°00	80°46	83°49
66°03	77°29	81°35	69°03	79°10	82°44	72°03	80°48	83°50
66°06	77°30	81°36	69°06	79°11	82°45	72°06	80°50	83°51
66°09	77°32	81°37	69°09	79°13	82°46	72°09	80°51	83°52
66°12	77°34	81°38	69°12	79°15	82°47	72°12	80°53	83°53
66°15	77°36	81°39	69°15	79°16	82°48	72°15	80°54	83°55
66°18	77°37	81°41	69°18	79°18	82°49	72°18	80°56	83°56
66°21	77°39	81°42	69°21	79°20	82°50	72°21	80°58	83°57
66°24	77°41	81°43	69°24	79°21	82°52	72°24	80°59	83°58
66°27	77°42	81°44	69°27	79°23	82°53	72°27	81°01	83°59
66°30	77°44	81°45	69°30	79°25	82°54	72°30	81°02	84°00
66°33	77°46	81°46	69°33	79°26	82°55	72°33	81°04	84°01
66°36	77°47	81°48	69°36	79°28	82°56	72°36	81°06	84°02
66°39	77°49	81°49	69°39	79°30	82°57	72°39	81°07	84°03
66°42	77°51	81°50	69°42	79°31	82°58	72°42	81°09	84°04
66°45	77°53	81°51	69°45	79°33	82°59	72°45	81°11	84°05
66°48	77°54	81°52	69°48	79°35	83°00	72°48	81°12	84°07
66°51	77°56	81°53	69°51	79°36	83°02	72°51	81°14	84°08
66°54	77°58	81°54	69°54	79°38	83°03	72°54	81°15	84°09
66°57	77°59	81°56	69°57	79°40	83°04	72°57	81°17	84°10
67°00	78°01	81°57	70°00	79°41	83°05	73°00	81°19	84°11
67°03	78°03	81°58	70°03	79°43	83°06	73°03	81°20	84°12
67°06	78°04	81°59	70°06	79°44	83°07	73°06	81°22	84°13
67°09	78°06	82°00	70°09	79°46	83°08	73°09	81°23	84°14
67°12	78°08	82°01	70°12	79°48	83°09	73°12	81°25	84°15
67°15	78°10	82°03	70°15	79°49	83°11	73°15	81°27	84°16
67°18	78°11	82°04	70°18	75°51	83°12	73°18	81°28	84°17
67°21	78°13	82°05	70°21	79°53	83°13	73°21	81°30	84°18
67°24	78°15	82°06	70°24	79°54	83°14	73°24	81°31	84°20
67°27	78°16	82°07	70°27	79°56	83°15	73°27	81°33	84°21
67°30	78°18	82°08	70°30	79°58	83°16	73°30	81°35	84°22
67°33	78°20	82°09	70°33	79°59	83°17	73°33	81°36	84°23
67°36	78°21	82°11	70°36	80°01	83°18	73°36	81°38	84°24
67°39	78°23	82°12	70°39	80°02	83°19	73°39	81°39	84°25
67°42	78°25	82°13	70°42	80°04	83°21	73°42	81°41	84°26
67°45	78°26	82°14	70°45	80°06	83°22	73°45	81°42	84°27
67°48	78°28	82°15	70°48	80°07	83°23	73°48	81°44	84°28
67°51	78°30	82°16	70°51	80°09	83°24	73°51	81°46	84°29
67°54	78°31	82°18	70°54	80°11	83°25	73°54	81°47	84°30
67°57	78°33	82°19	70°57	80°12	83°26	73°57	81°49	84°31
68°00	78°35	82°20	71°00	80°14	83°27	74°00	81°50	84°32

1 (74°00—83°00).	2 (81°50—86°29).	3 (84°32—87°39).	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3
74°00	81°50	84°32	77°00	83°25	85°36	80°00	84°58	86°38
74°03	81°52	84°33	77°03	83°26	85°37	80°03	84°59	86°39
74°06	81°54	84°35	77°06	83°28	85°38	80°06	85°01	86°40
74°09	81°55	84°36	77°09	83°30	85°39	80°09	85°02	86°41
74°12	81°57	84°37	77°12	83°31	85°40	80°12	85°04	86°42
74°15	81°58	84°38	77°15	83°33	85°41	80°15	85°05	86°43
74°18	82°00	84°39	77°18	83°34	85°42	80°18	85°07	86°44
74°21	82°02	84°40	77°21	83°36	85°43	80°21	85°08	86°45
74°24	82°03	84°41	77°24	83°37	85°44	80°24	85°10	86°46
74°27	82°05	84°42	77°27	83°39	85°45	80°27	85°11	86°47
74°30	82°06	84°43	77°30	83°40	85°46	80°30	85°13	86°48
74°33	82°08	84°44	77°33	83°42	85°47	80°33	85°15	86°49
74°36	82°10	84°45	77°36	83°44	85°48	80°36	85°16	86°50
74°39	82°11	84°46	77°39	83°45	85°50	80°39	85°18	86°52
74°42	82°13	84°47	77°42	83°47	85°51	80°42	85°19	86°53
74°45	82°14	84°48	77°45	83°49	85°52	80°45	85°21	86°54
74°48	82°16	84°50	77°48	83°50	85°53	80°48	85°22	86°55
74°51	82°17	84°51	77°51	83°52	85°54	80°51	85°24	86°56
74°54	82°19	84°52	77°54	83°53	85°55	80°54	85°25	86°57
74°57	82°21	84°53	77°57	83°54	85°56	80°57	85°27	86°58
75°00	82°22	84°54	78°00	83°56	85°57	81°00	85°28	86°59
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75°30	82°38	85°04	78°30	84°11	86°07	81°30	85°44	87°09
75°33	82°39	85°05	78°33	84°13	86°08	81°33	85°45	87°10
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75°39	82°43	85°08	78°39	84°16	86°10	81°39	85°48	87°12
75°42	82°44	85°09	78°42	84°18	86°11	81°42	85°50	87°13
75°45	82°46	85°10	78°45	84°19	86°12	81°45	85°51	87°14
75°48	82°47	85°11	78°48	84°21	86°13	81°48	85°53	87°15
75°51	82°49	85°12	78°51	84°22	86°15	81°51	85°54	87°16
75°54	82°50	85°13	78°54	84°24	86°16	81°54	85°56	87°17
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76°09	82°58	85°18	79°09	84°32	86°21	82°09	86°03	87°22
76°12	83°00	85°19	79°12	84°33	86°22	82°12	86°05	87°23
76°15	83°01	85°20	79°15	84°35	86°23	82°15	86°06	87°24
76°18	83°03	85°21	79°18	84°36	86°24	82°18	86°08	87°25
76°21	83°05	85°22	79°21	84°38	86°25	82°21	86°09	87°26
76°24	83°06	85°23	79°24	84°39	86°26	82°24	86°11	87°27
76°27	83°08	85°24	79°27	84°41	86°27	82°27	86°13	87°28
76°30	83°09	85°25	79°30	84°42	86°28	82°30	86°14	87°29
76°33	83°11	85°27	79°33	84°44	86°29	82°33	86°16	87°30
76°36	83°12	85°28	79°36	84°45	86°30	82°36	86°17	87°31
76°39	83°14	85°29	79°39	84°47	86°31	82°39	86°19	87°32
76°42	83°16	85°30	79°42	84°48	86°32	82°42	86°20	87°33
76°45	83°17	85°31	79°45	84°50	86°33	82°45	86°22	87°34
76°48	83°19	85°32	79°48	84°52	86°34	82°48	86°23	87°35
76°51	83°20	85°33	79°51	84°53	86°35	82°51	86°25	87°36
76°54	83°22	85°34	79°54	84°55	86°36	82°54	86°26	87°37
76°57	83°23	85°35	79°57	84°56	86°37	82°57	86°28	87°38
77°00	83°25	85°36	80°00	84°58	86°38	83°00	86°29	87°39

1	2	3	4	5	6	7	8	9
5	10	15	19½	23½	27½	31½	35	38
5½	11	16	21	25½	30	34	37½	41
6	12	17½	23	27½	32	36½	40	43½
6½	13	19	24½	29½	34½	38½	42½	45½
7	14	20	26	31½	36½	40½	44½	48
7½	15	21½	28	33½	38½	42½	46½	50
8	15½	23	29½	35	40	44½	48½	51½
8½	16½	24	31	37	42	46½	50	53½
9	17½	25½	32½	38½	43½	48	51½	55
9½	18½	26½	34	40	45	49½	53	56½
10	19½	28	35	41½	46½	51	54½	58
10½	20½	29	36½	43	48	52½	56	59
11	21	30	38	44	49½	53½	57½	60
11½	22	31½	39	45½	50½	55	58½	61½
12	23	32½	40½	46½	52	56	59½	62½
12½	24	33½	41½	48	53	57	60½	63½
13	25	34½	42½	49	54	58½	61½	64½
13½	25½	36	44	50	55	59	62½	65
14	26½	37	45	51½	56	60	63½	66
14½	27½	38	46	52½	57	61	64	67
15	28	39	47	53½	58	62	65	67½
15½	29	40	48	54	59	62½	65½	68
16	30	40½	49	55	60	63½	66½	69
16½	30½	41½	50	56	60½	64½	67	69½
17	31½	42½	50½	57	61½	65	68	70
17½	32	43	51½	57½	62	65½	68½	70½
18	33	44½	52½	58½	63	66½	69	71
18½	34	45	53	59	63½	67	69½	71½
19	34½	46	54	60	64	67½	70	72
19½	35½	46½	55	60½	65	68	70½	72½
20	36	47½	55½	61	65½	68½	71	73
20½	37	48½	56	62	66	69	71½	73½
21	37½	49	57	62½	66½	69½	72	74
21½	38	50	57½	63	67	70	72½	74
22	39	50½	58½	63½	67½	70½	73	74½
22½	39½	51	59	64	68	71	73	75
23	40½	52	59½	65	68½	71½	73½	75¼
23½	41	52½	60	65½	69	72	74	75¾
24	41½	53	60½	66	69½	72	74½	76
24½	42½	54	61½	66½	70	72½	74½	76¼
25	43	54½	62	67	70½	73	75	76½
25½	43½	55	62½	67½	70½	73½	75¼	77
26	44½	55½	63	67½	71	73½	75½	77¼
26½	45	56	63½	68	71½	74	76	77½
27	45½	57	64	68½	72	74½	76¼	77¾
27½	46	57½	64½	69	72	74½	76½	78
28	47	58	65	69½	72½	75	76¾	78¼
28½	47½	58½	65½	70	73	75¼	77	78½
29	48	59	65½	70	73½	75½	77¼	78¾
29½	48½	59½	66	70½	73½	75¾	77½	79
30	49	60	66½	71	74	76	77¾	79
30½	49½	60½	67	71	74	76¼	78	79¼
31	50	61	67½	71½	74½	76½	78¼	79½
31½	51	61½	68	72	75	77	78½	79¾
32	51½	62	68	72½	75	77	78¾	80
32½	52	62½	68½	72½	75¼	77½	79	80
33	52½	63	69	73	75½	77½	79	80¼
33½	53	63½	69½	73	75¾	77¾	79¼	80½
34	53½	63½	69½	73½	76	78	79½	80¾
34½	54	64	70	74	76¼	78¼	79¾	81
35	54½	64½	70½	74	76½	78½	79¾	81

1	2	3	4	5	6	7	8	9
35	54½	64½	70½	74	76½	78½	79¾	81
35½	55	65	70½	74½	76¾	78¾	80	81¼
36	55½	65½	71	74½	77	79	80¼	81¼
36½	56	65½	71½	75	77¼	79	80½	81½
37	56½	66	71½	75	77½	79¼	80½	81½
37½	57	66½	72	75½	77¾	79½	80¾	81¾
38	57½	67	72½	75¾	78	79¾	81	82
38½	58	67½	72½	76	78¼	79¾	81	82
39	58½	67½	73	76¼	78¼	80	81¼	82¼
39½	59	68	73	76¼	78½	80¼	81¼	82¼
40	59	68½	73½	76½	78¾	80¼	81½	82½
40½	59½	68½	73½	76¾	79	80½	81¾	82½
41	60	69	74	77	79¼	80½	81¾	82¾
41½	60½	69½	74	77¼	79¼	80¾	82	82¾
42	61	69½	74½	77½	79½	81	82	83
42½	61½	70	74½	77½	79¾	81¼	82¼	83
43	62	70½	75	78	79¾	81¼	82½	83¼
43½	62	70½	75¼	78	80	81½	82½	83¼
44	62½	71	75½	78¼	80¼	81½	82½	83½
44½	63	71½	75¾	78½	80¼	81¾	82¾	83½
45	63½	71½	76	78¾	80½	81¾	82¾	83¾
45½	64	72	76¼	79	80¾	82	83	83¾
46	64	72	76½	79	80¾	82¼	83	83¾
46½	64½	72½	76¾	79¼	81	82¼	83¼	84
47	65	72½	77	79½	81¼	82½	83¼	84
47½	65½	73	77	79½	81¼	82½	83½	84¼
48	66	73½	77¼	79¾	81½	82¾	83½	84¼
48½	66	73½	77½	80	81½	82¾	83¾	84½
49	66½	74	77¾	80¼	81¾	83	83¾	84½
49½	67	74	78	80¼	82	83	84	84½
50	67	74½	78¼	80½	82	83¼	84	84¾
50½	67½	74½	78¼	80¾	82¼	83¼	84	84¾
51	68	75	78½	80¾	82¼	83½	84¼	84¾
51½	68½	75¼	78¾	81	82½	83½	84¼	85
52	68½	75½	79	81	82½	83¾	84½	85
52½	69	75¾	79¼	81¼	82¾	83¾	84½	85
53	69½	76	79¼	81½	82¾	83¾	84½	85¼
53½	69½	76¼	79½	81½	83	84	84¾	85¼
54	70	76½	79½	81¾	83	84	84¾	85½
54½	70½	76¾	80	82	83¼	84¼	85	85½
55	70½	76¾	80	82	83¼	84¼	85	85½
55½	71	77	80¼	82¼	83½	84½	85	85¾
56	71½	77¼	80½	82¼	83½	84½	85¼	85¾
56½	71½	77½	80½	82½	83¾	84½	85¼	85¾
57	72	77¾	80¾	82½	83¾	84¾	85¼	85¾
57½	72½	78	81	82¾	84	84¾	85½	86
58	72½	78¼	81	83	84	85	85½	86
58½	73	78½	81¼	83	84¼	85	85½	86
59	73½	78½	81½	83¼	84¼	85	85¾	86¼
59½	73½	79	81½	83¼	84½	85¼	85¾	86¼
60	74	79	81¾	83½	84½	85¼	85¾	86½
60½	74	79¼	82	83½	84½	85½	86	86½
61	74½	79½	82¼	83¾	84¾	85½	86	86½
61½	75	79¾	82¼	83¾	84¾	85½	86	86½
62	75	80	82½	84	85	85¾	86¼	86½
62½	75½	80¼	82½	84	85	85¾	86¼	86¾
63	75¾	80¼	82¾	84¼	85¼	85¾	86¼	86¾
63½	76	80½	83	84¼	85¼	86	86½	86¾
64	76¼	80¾	83	84½	85¼	86	86½	87
64½	76½	81	83¼	84½	85½	86	86½	87
65	76¾	81¼	83¼	84¾	85½	86¼	86¾	87

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